XVI. *On the Method of Symmetric Products, and on Certain Circular Functions connected with that Method. By the Rev.* Robert Harley, *F.R.A.S., Corresponding Member of the Literary and Philosophical Society of Manchester. Communicated by* A. Cayley, *Esq., F.R.S.*

Received October 18,—Read December 13, 1860.

In a paper printed in the second part of the fifteenth volume of the 'Manchester Memoirs,' I have given a systematic exposition of Mr. Cockle's Method of Symmetric Products, and its application to the finite algebraic solution of the lower equations. In that paper, to which I shall in future refer as my original memoir, I have also defined a new cyclical symbol, and I have by its aid succeeded in effecting the direct calculation of a certain sextic equation, on whose solution that of the general quintic may be made to depend. In an Addendum I have pointed out the connexion between the circular functions which occur in my own researches and those to which we are led by the theory of Lagrange and Vandermonde, and, by means of the cyclical process, I have given a neat expression for the first coefficient of Lagrange's reducing equation. These researches I have followed up in an article "On the Theory of Quintics," in the third volume of the 'Quarterly Journal of Pure and Applied Mathematics.' My present purpose is not to repeat, but to endeavour to generalize and extend former results. I shall therefore content myself with a very brief *résumé* of my investigations, referring the reader for details to the above works. Mr. Cockle's earlier researches on the subject were published in a series of five papers "On the Transformation of Algebraic Equations," printed in the first and third volumes of 'The Mathematician*.'

Section I.

The Method of Symmetric Products, and a New Application of it to the Solution of the Lower Equations.

1. Any n symbols $x_1, x_2, x_3, \dots x_n$ may be regarded as the roots of an equation of the form

$$(a, b, c, \dots \between x, 1)^n = a(x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)=0 \dagger.$$

* The first paper of the series appeared in 'The Mathematician' for March 1844. Mr. Cockle's subsequent contributions to the subject will be found in the 'Mechanics' Magazine,' the 'Cambridge and Dublin Mathematical Journal,' the 'Lady's and Gentleman's Diary,' the 'Philosophical Magazine,' 'Liouville's Journal,' the 'Quarterly Journal of Pure and Applied Mathematics,' the 'Manchester Memoirs,' and elsewhere.

† Throughout the whole of this paper I adopt Mr. Cayley's quantical notation. In the ordinary nota-

Let $X_1, X_2, X_3, \ldots X_{n-1}$ be linear unsymmetric functions of x and of the form

$$x_1+k_1x_2+k_2x_3+\ldots+k_{n-2}x_{n-1}+k_{n-1}x_n,$$

where the $n-1$ constants $k_1, k_2, \ldots k_{n-1}$ are arbitrary. Then if n be less than 5, the constants which occur in the $n-1$ functions may be so distributed and determined as to render the product

$$\pi_{n-1}(x) \text{ or } X_1X_2X_3\ldots X_{n-1}$$

(or, when $n=2$, X^2) symmetric relatively to x; but if n be equal to, or greater than 5, the symmetry is not in general attainable. The product $\pi_{n-1}(x)$ is called the *symmetric* or *resolvent product* according as it is or is not symmetric. When this symmetry exists, $\pi_{n-1}(x)$ can of course be expressed as a rational function of the coefficients a, b, c, &c. When it does not exist, we form the analogues of the functions which enter into the symmetric cases, and the symmetric product $\Pi(x)$ is then obtained by multiplying together the several unequal values which the resolvent product $\pi_{n-1}(x)$ can be made to take by the permutation of the x's; and $\Pi(x)$ may, in this case also, be expressed as a rational function of the coefficients.

2. The simplest case is the quadratic

$$(a, b, c\between x, 1)^2=a(x-x_1)(x-x_2)=0.$$

Here

$$\{\pi_1(x)\}^2=X^2=(x_1+kx_2)^2,$$

and it is seen at a glance that when $k=-1$, X is unsymmetric and X^2 is symmetric. Hence

$$X^2=(\Sigma x)^2-4x_1x_2=\frac{1}{a^2}(b^2-4ac),$$

and the roots of the quadratic are

$$\frac{1}{2a}\{-b\pm a\pi_1(x)\}=\frac{1}{2a}(-b\pm\sqrt{b^2-4ac}).$$

3. The next case is the cubic

$$(a, b, c, d\between x, 1)^3=a(x-x_1)(x-x_2)(x-x_3)=0.$$

Assume

$$X_1=x_1+k_1x_2+k_2x_3,$$

$$X_2=x_1+k_2x_2+k_1x_3,$$

and

$$\pi_2(x)=X_1X_2;$$

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tion $(a, b, c, \ldots\between x, y)^n$ would be written

$$ax^n+bx^{n-1}y+cx^{n-2}y^2+\ \&c.,$$

and $(a, b, c, \ldots\between x, y)^n$ would be written

$$ax^n+\frac{n}{1}bx^{n-1}y+\frac{n(n-1)}{1.2}cx^{n-2}y^2+\&c.$$

then, combining the conditions of symmetry, and rejecting those values of k (k_1 or k_2 indifferently) which render X symmetric, we find

$$k^2+k+1=0,$$

that is, k is an unreal cube root of unity. Represent this root by α, and put

$$f(\alpha)=x_1+\alpha\, x_2+\alpha^2x_3;$$

then

$$f(\alpha^2)=x_1+\alpha^2x_2+\alpha\, x_3,$$

and

$$\pi_2(x)=f(\alpha).f(\alpha^2)=(\Sigma x)^2-3\Sigma x_1x_2=\frac{1}{a^2}(b^2-3ac).$$

If, in evolving the roots by this method, we replace $f(\alpha)$ by f, and $\pi_2(x)$ by π, there will result

$$f=x_1+\alpha\, x_2+\alpha^2x_3,$$

$$\frac{\pi}{f}=x_1+\alpha^2x_2+\alpha\, x_3,$$

$$-\frac{b}{a}=x_1+\ x_2+\ x_3;$$

so that

$$f+\frac{\pi}{f}=\frac{1}{a}(b+3ax_1),$$

$$\alpha^2f+\frac{\pi}{\alpha^2f}=\frac{1}{a}(b+3ax_2),$$

$$\alpha f+\frac{\pi}{\alpha f}=\frac{1}{a}(b+3ax_3);$$

and therefore

$$f^3+\frac{\pi^3}{f^3}=\frac{1}{a^3}(1,\ -3b,\ 3^2ac,\ -3^3a^2d \between b,\ 1)^3$$

$$=\frac{1}{a^3}(-2b^3+9abc-27a^2d);$$

whence, solving as for a quadratic in f^3, restoring the value of π, and extracting the cube root, we have

$$f=\frac{1}{a\sqrt[3]{2}}\left\{-27a^2d+9abc-2b^3+3a\sqrt{3}(27a^2d^2-18abcd+4ac^3+4b^3d-b^2c^2)^{\frac{1}{2}}\right\}^{\frac{1}{3}},$$

and the roots of the complete cubic are included in the formula

$$\frac{1}{3}\left(\alpha^m f+\frac{\pi}{\alpha^m f}-\frac{b}{a}\right),$$

where $m=1$, 2, or 3.

In my original memoir I followed Mr. COCKLE, and in the application of the theory to the solution of the cubic and the biquadratic, I employed a subsidiary equation of the same degree. This equation was obtained by eliminating x between the given one and

$$y-\psi(x)=0,$$

ψ being rational and so constructed as to make $\pi(y)$ vanish. It is true that the evanescence of π leads to an immediate solution, and that when y is known, x is also known. But this evanescence is not essential to the theory; and we are conducted to more significant results by dispensing with it.

4. For the quartic

$$(a, b, c, d, e \between x, 1)^4=0,$$

assume (cyclically)

$$X_1=x_1+k_1x_2+k_2x_3+k_3x_4,$$
$$X_2=x_1+k_2x_2+k_3x_3+k_1x_4,$$
$$X_3=x_1+k_3x_2+k_1x_3+k_2x_4,$$

and

$$\pi_3(x)=X_1\,X_2\,X_3.$$

Then, as before, combining the conditions of symmetry and rejecting those values of k which would render X symmetric, we are led to the cubic

$$k^3+k^2-k-1=0,$$

of which the roots are -1, 1 and -1. Let therefore

$$X_1=x_1-x_2+x_3-x_4,$$
$$X_2=x_1+x_2-x_3-x_4,$$
$$X_3=x_1-x_2-x_3+x_4,$$

then

$$\pi_3(x)=X_1X_2X_3=(\Sigma x)^3-4\Sigma x\Sigma x_1x_2+8\Sigma x_1x_2x_3=\frac{1}{a^3}(-b^3+4abc-8a^2d).$$

5. The following solution is due to Mr. COCKLE, who communicated it to me in September of last year. It may be considered as an extension of a solution of the complete cubic which I sent to him in January of the same year, and which, not essentially differing from the above, has a certain resemblance to that given by MURPHY in the 'Philosophical Transactions' for 1837.

Let

$$f_1=x_1-x_2+x_3-x_4,$$

and

$$f_2=x_1+x_2-x_3-x_4;$$

then

$$\frac{\pi}{f_1f_2}=x_1-x_2-x_3-x_4,$$

and as in cubics,

$$f_1+f_2-\frac{\pi}{f_1f_2}=\frac{1}{a}(b+4ax_1),$$

$$-f_1+f_2-\frac{\pi}{f_1f_2}=\frac{1}{a}(b+4ax_2),$$

$$f_1-f_2-\frac{\pi}{f_1f_2}=\frac{1}{a}(b+4ax_3),$$

$$-f_1-f_2+\frac{\pi}{f_1f_2}=\frac{1}{a}(b+4ax_4);$$

therefore

$$\left(f_1^2+f_2^2-\frac{\pi^2}{f_1^2f_2^2}\right)^2-4f_1^2f_2^2$$

$$=\frac{1}{a^4}(1,\ -4b,\ 4^2ac,\ -4^3a^2d,\ 4^4a^3e \between b,\ 1)^4$$

$$=\frac{1}{a^4}(256a^3e-64a^2bd+16ab^2c-3b^4);$$

and

$$f_1^2+f_2^2+\frac{\pi^2}{f_1^2f_2^2}=\frac{1}{a^2}(b^2+2ab\Sigma x+4a^2\Sigma x^2)$$

$$=\frac{1}{a^2}(-8ac+3b^2);$$

consequently

$$\left(f_1^2+f_2^2+\frac{\pi^2}{f_1^2f_2^2}\right)^2-\left(f_1^2+f_2^2-\frac{\pi^2}{f_1^2f_2^2}\right)^2+4f_1^2f_2^2$$

$$=\frac{1}{a^4}\{(8ac-3b^2)^2-(256a^3e-64a^2bd+16ab^2c-3b^4)\};$$

or

$$f_1^2f_2^2+f_1^2\cdot\frac{\pi^2}{f_1^2f_2^2}+f_2^2\cdot\frac{\pi^2}{f_1^2f_2^2}=\frac{1}{a^4}(-64a^3e+16a^2bd+16a^2c^2-16ab^2c+3b^4).$$

It hence appears that f_1^2, f_2^2 and $\frac{\pi^2}{f_1^2f_2^2}$ are the roots of the cubic

$$(a^4,\ 8a^3c-3a^2b^2,\ -64a^3e+16a^2bd+16a^2c^2-16ab^2c+3b^4,\ a^4\pi^2 \between f^2,\ 1)^3=0.$$

When f is known, x is given by

$$x_1=\tfrac{1}{4}\left(f_1+f_2+\frac{\pi}{f_1f_2}-\frac{b}{a}\right),$$

and the corresponding formulæ for x_2, x_3, x_4 may be readily obtained.

6. Next, for the quintic

$$(a,\ b,\ c,\ d,\ e,\ f \between x,\ 1)^5=0,$$

assume

$$X_1=x_1+k_1x_2+k_2x_3+k_3x_4+k_4x_5,$$
$$X_2=x_1+k_2x_2+k_4x_3+k_1x_4+k_3x_5,$$
$$X_3=x_1+k_3x_2+k_1x_3+k_4x_4+k_2x_5,$$
$$X_4=x_1+k_4x_2+k_3x_3+k_2x_4+k_1x_5,$$

and $\pi_4(x)=X_1X_2X_3X_4$.

In regard to the above distribution of the constants k_1, k_2, k_3, k_4, it will be observed that they are arranged according to the following scheme:

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

That is, the four horizontal rows read downwards are identical in value and order with the four vertical columns read from left to right, while k_1 and k_4 lie in inverse symmetry upon, and k_2 and k_3 around, diagonals. Probably no other distribution would render $\pi_4(x)$ more nearly symmetrical*.

Combining, as in former cases, the conditions of symmetry and rejecting incongruous results, we arrive at the quartic

$$k^4+k^3+k^2+k+1=0,$$

of which the roots are the unreal fifth roots of unity. Let then ω, ω^2, ω^3, and ω^4 denote these roots, and let

$$f(\omega)=x_1+\omega x_2+\omega^2x_3+\omega^3x_4+\omega^4x_5;$$

then

$$f(\omega^2)=x_1+\omega^2x_2+\omega^4x_3+\omega x_4+\omega^3x_5,$$
$$f(\omega^3)=x_1+\omega^3x_2+\omega x_3+\omega^4x_4+\omega^2x_5,$$
$$f(\omega^4)=x_1+\omega^4x_2+\omega^3x_3+\omega^2x_4+\omega x_5,$$

and

$$\begin{aligned}\pi_4(x)&=f(\omega).f(\omega^2).f(\omega^3).f(\omega^4)\\ &=(\Sigma x)^4-5(\Sigma x)^2\Sigma x_1x_2+5(\Sigma x_1x_2)^2+5\tau\tau'\\ &=\frac{1}{a^4}(b^4-5ab^2c+5a^2c^2)+5\tau\tau',\end{aligned}$$

where

$$\tau=x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1,$$

and

$$\tau'=x_1x_3+x_2x_4+x_3x_5+x_4x_1+x_5x_2.$$

These two functions, τ and τ', are circular, and complementary to each other. Since their product is unsymmetric relatively to x, it follows that $\pi_4(x)$ cannot in general be rendered symmetric. Before proceeding to discuss this product, it will be convenient to introduce some other matters connected with the general theory.

SECTION II.—*Circular Functions and the New Cyclical Symbol.*

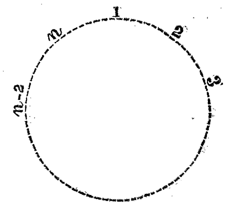
7. In the transformation and general treatment of the higher equations circular functions occupy a conspicuous place, and play an important part. An attentive consideration of the structure of such functions will enable us to devise a calculus whereby operations upon them will be materially abridged. The theory is far from being complete, and its practical application admits of great improvement. In my original memoir I have proposed and applied a symbol which not only helps, I think, to throw some light on the general theory, but also enables us to effect with ease and rapidity calculations which would otherwise be very laborious, if not wholly impracticable. The method there employed exhibits to the eye and to the mind the various combinations of dimen-

* When all the divisors of n are even, a cyclical arrangement must be adopted, as in the case for quartics (art. 4). When n is prime or odd, an arrangement similar to the above, or a modification of it, will probably be found available.

sions without rendering it necessary to examine all the results of multiplication, or to make a hazardous selection of those which may be deemed material. Combining as many terms (n) in one as there are roots involved, the cyclical symbol reduces the labour of multiplication alone by $\frac{n-1}{n}$th; and in passing from the circular to the corresponding symmetric function the process also affords considerable facilities.

8. Let $\chi(0)$ be a function of x, and let $\chi(q)$ be derived from $\chi(0)$ by advancing each of the roots contained in its q steps in a given cycle r. *Ex. gr.* Suppose that

$$\chi(0)=x_1x_2,$$



and that we are following the first cycle, which, indeed, may be taken as a type of all others, and is found in actual practice to be the most easy with which to operate. Then, according to the definition,

$$\begin{aligned}
&\chi(1)=x_2x_3,\\
&\chi(2)=x_3x_4,\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\chi(n-1)=x_nx_1,\\
&\chi(n) \quad =x_1x_2=\chi(0),\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Or again, if we suppose $n=5$, and

$$\chi(0)=x_1^2(x_2x_5+x_3x_4),$$

then, following the same cycle, we have

$$\begin{aligned}
&\chi(1)=x_2^2(x_3x_1+x_4x_5),\\
&\chi(2)=x_3^2(x_4x_2+x_5x_1),\\
&\chi(3)=x_4^2(x_5x_3+x_1x_2),\\
&\chi(4)=x_5^2(x_1x_4+x_2x_3),\\
&\chi(5)=x_1^2(x_2x_5+x_3x_4)=\chi(0),\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

and so on.

Next, let $\Sigma'_r\chi(0)$ or, more simply, $\Sigma'\chi(0)$* represent the circular function

$$\chi(0)+\chi(1)+\ldots+\chi(n-2)+\chi(n-1).$$

Then, since each root recurs at every nth step in the cycle, we have

$$\begin{aligned}
&\chi(n) \quad =\chi(0),\\
&\chi(n+1)=\chi(1),\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\chi(n+q)=\chi(q);
\end{aligned}$$

* When there is only one cycle involved in the operation, or when there is no comparison of cycles, it is not necessary to suffix Σ'.

and therefore

$$\begin{aligned}\Sigma'\chi(1) &= \chi(1)+\chi(2)+\ldots+\chi(n-1)+\chi(n)\\ &=\{\chi(1)+\chi(2)+\ldots+\chi(n-1)\}+\chi(0)\\ &=\Sigma'\chi(0).\end{aligned}$$

Similarly,

$$\Sigma'\chi(2)=\Sigma'\chi(1)=\Sigma'\chi(0),$$

and by generalization and induction,

$$\Sigma'\chi(q)=\Sigma'\chi(0).$$

Hence

THEOREM I.—A circular function is not affected in value by the simultaneous advancing or receding of the roots which it contains any number of steps in the cycle to which it belongs.

Whence it follows that $\Sigma'\chi_1(0).\Sigma'\chi_2(0)$, or its equivalent,

$$\{\chi_1(0)+\chi_1(1)+\ldots+\chi_1(n-2)+\chi_1(n-1)\}.\Sigma'\chi_2(0),$$

may be written

$$\chi_1(0).\Sigma'\chi_2(0)+\chi_1(1).\Sigma'\chi_2(1)+\ldots+\chi_1(n-2).\Sigma'\chi_2(n-2)+\chi_1(n-1).\Sigma'\chi_2(n-1),$$

which is equal to

$$\Sigma'\{\chi_1(0).\Sigma'\chi_2(0)\};$$

and by simply interchanging χ_1 and χ_2, we have also

$$\Sigma'\chi_1(0).\Sigma'\chi_2(0)=\Sigma'\{\chi_2(0).\Sigma'\chi_1(0)\}.$$

Which gives us

THEOREM II.—The product of two circular functions belonging to the same cycle is itself a circular function to that cycle, and is given by the application of the cyclical symbol to the product of either function into the initial or leading terms of the other.

It should be remarked that Σ' is a symbol of cyclical operation, and subject to the same laws, with certain obvious limitations as if it were a symbol of quantity. Thus

$$\Sigma'\{\chi_1(0)+\chi_2(0)+\text{ \&c.}\}=\Sigma'\chi_1(0)+\Sigma'\chi_2(0)+\text{ \&c.};$$

and, in general, if we develope

$$\{\chi_1(0)+\chi_2(0)+\text{\&c.}\}^m$$

by the multinomial theorem, and then apply the symbol Σ' to each term, the result will be equal to

$$\Sigma'\{\chi_1(0)+\chi_2(0)+\text{ \&c.}\}^m.$$

It should also be remarked that if C be a function or quantity such that it is not affected by the cyclical interchange of the roots, as when it is a constant quantity and therefore independent of the roots altogether, or as when it is a circular function and belongs to the same cycle as $\Sigma'\chi(0)$, then will

$$\Sigma'C\chi(0)=C\Sigma'\chi(0).$$

The foregoing theorems are true for any circular functions whatever, whether rational

or irrational, integral or fractional. But the following holds only for those circular functions which are rational and integral.

THEOREM III. If X be a circular function of the form

$$\Sigma'\{\chi_{\mathrm{a}}+\chi_{\mathrm{b}}+\chi_{\mathrm{c}}+\&c.\},$$

χ being defined by

$$\chi_{\mathrm{m}}=\mathfrak{a}'x_1^{\alpha}x_2^{\beta}x_3^{\gamma}\ldots x_n^{\xi}+\mathfrak{a}''x_1^{\beta}x_2^{\alpha}x_3^{\gamma}\ldots x_n^{\xi}+\mathfrak{a}'''\ \&c.,$$

where α, β, γ, ... ξ are positive integers or (some of them) zero, and m is the number of values of $x_1^{\alpha}x_2^{\beta}x_3^{\gamma}\ldots x_n^{\xi}$ or (what is the same thing) the number of terms contained in $\Sigma x_1^{\alpha}x_2^{\beta}x_3^{\gamma}\ldots x_n^{\xi}$, then will

$$\Sigma\mathrm{X}=nn'\left(\frac{1}{\mathrm{a}}\Sigma\chi_{\mathrm{a}}+\frac{1}{\mathrm{b}}\Sigma\chi_{\mathrm{b}}+\frac{1}{\mathrm{c}}\Sigma\chi_{\mathrm{c}}+\&c.\right),$$

n' being the number of values of X, and $\Sigma\chi_{\mathrm{m}}$ being of the form

$$(\mathfrak{a}'+\mathfrak{a}''+\mathfrak{a}'''+\&c.)\Sigma x_1^{\alpha}x_2^{\beta}x_3^{\gamma}\ldots x_n^{\xi}.$$

For if, as we are permitted, we fix one of the roots, and permute the remaining $n-1$ roots in all possible ways, there will arise $1.2.3\ldots(n-1)$, or (say) p, corresponding cycles; and if, for the moment, we represent by $\dot{\Sigma}$ the sum of the p expressions formed by applying these cycles to any one of the values of X, then, since Σ' consists of n, and consequently $\dot{\Sigma}\Sigma'$ of np, expressions of the form

$$\chi_{\mathrm{a}}+\chi_{\mathrm{b}}+\chi_{\mathrm{c}}+\&c.,$$

and since also $m=np$, or a submultiple of np, it follows that

$$\dot{\Sigma}\mathrm{X}=\frac{np}{\mathrm{a}}\Sigma\chi_{\mathrm{a}}+\frac{np}{\mathrm{b}}\Sigma\chi_{\mathrm{b}}+\frac{np}{\mathrm{c}}\Sigma\chi_{\mathrm{c}}+\&c.$$

But $\dot{\Sigma}\mathrm{X}=\frac{p}{n'}\Sigma\mathrm{X}$. Whence the theorem.

9. The symbol Σ' admits of an easy extension to functions of the form

$$f^m(\varrho)=x_1^m+\varrho x_2^m+\varrho^2x_3^m+\ldots+\varrho^{n-2}x_{n-1}^m+\varrho^{n-1}x_n^m\,*,$$

ϱ being an nth root of unity, real or imaginary. For, if we represent this function by $\Sigma'_{\varrho}x^m$, we shall have

$$f^m(\varrho^q).f^p(\varrho^{n-q})=\Sigma'\{x^mf^p(\varrho^{n-q})\}=\Sigma'\{x^pf^m(\varrho^q)\},$$

or

$$\Sigma'_{\varrho^q}x^m.\Sigma'_{\varrho^{n-q}}x^p=\Sigma'(x^m\Sigma'_{\varrho^{n-q}}x^p)=\Sigma'(x^p\Sigma'_{\varrho^q}x^m),$$

where Σ' is the simple or ordinary cyclical function to the first cycle

$$\ldots123\ldots(n-1)123\ldots(n-1)\ldots$$

This may be proved thus:—

$$f^m(\varrho^q)=x_1^m+\varrho^qx_2^m+\varrho^{2q}x_3^m+\ldots+\varrho^{q(n-2)}x_{n-1}^m+\varrho^{q(n-1)}x_n^m=\Sigma'_{\varrho^q}x^m,$$

and

$$f^p(\varrho^{n-q})=x_1^p+\varrho^{n-q}x_2^p+\varrho^{2n-2q}x_3^p+\ldots+\varrho^{(n-q)(n-2)}x_{n-1}^p+\varrho^{(n-q)(n-1)}x_n^p=\Sigma'_{\varrho^{n-q}}x^p;$$

* The idea of extending Σ' to functions of this form was suggested to me by Mr. COCKLE, in a letter under date January 8, 1859.

whence, multiplying, bearing in mind that $\varrho^n=1$, and arranging the multiplicand as below,

$$\begin{aligned}
&x_1^p(x_1^m+\varrho^q x_2^m+\varrho^{2q}x_3^m+\ldots+\varrho^{q(n-2)}x_{n-1}^m+\varrho^{q(n-1)}x_n^m)\\
&+x_2^p(x_2^m+\varrho^q x_3^m+\varrho^{2q}x_4^m+\ldots+\varrho^{q(n-2)}x_n^m\ \ +\varrho^{q(n-1)}x_1^m)\\
&+x_3^p(x_3^m+\varrho^q x_4^m+\varrho^{2q}x_5^m+\ldots+\varrho^{q(n-2)}x_1^m\ \ +\varrho^{q(n-1)}x_2^m)\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&+x_n^p(x_n^m+\varrho^q x_1^m+\varrho^{2q}x_2^m+\ldots+\varrho^{q(n-2)}x_{n-2}^m+\varrho^{q(n-1)}x_{n-1}^m),
\end{aligned}$$

we see that $f^m(\varrho^q).f^p(\varrho^{n-q})$, or

$$\begin{aligned}
\Sigma'_{\varrho^q}x^m\,.\,\Sigma'_{\varrho^{n-q}}x^p&=\Sigma'\{x_1^p(x_1^m+\varrho^q x_2^m+\varrho^{2q}x_3^m+\ldots+\varrho^{q(n-2)}x_{n-1}^m+\varrho^{q(n-1)}x_n^m)\}\\
&=\Sigma'(x^p\Sigma'_{\varrho^q}x^m),
\end{aligned}$$

which establishes one part of the theorem; and the other part is established by simply interchanging m and p.

10. In order to illustrate some of the preceding properties of Σ', let us take one of the factors

$$f(\alpha)=x_1+\alpha x_2+\alpha^2 x_3,$$

which enters into the symmetric product for cubics (art. 3). Then $\pi_2(x)$, or

$$\begin{aligned}
f(\alpha).f(\alpha^2)&=\Sigma'\{x_1(x_1+\alpha x_2+\alpha^2x_3)\}\\
&=\Sigma x^2+(\alpha+\alpha^2)\Sigma' x_1x_2\\
&=\Sigma x^2-\Sigma x_1x_2.
\end{aligned}$$

Next, let us take the factor

$$f(\omega)=x_1+\omega x_2+\omega^2x_3+\omega^3x_4+\omega^4x_5,$$

which enters into the resolvent product for quintics (art. 6). Then $\pi_4(x)$, or

$$\begin{aligned}
&\{f(\omega).f(\omega^4)\}\times\{(f(\omega^2).f(\omega^3)\}\\
&\quad=\Sigma'\{x_1(x_1+\omega\, x_2+\omega^2x_3+\omega^3x_4+\omega^4x_5)\}\times\\
&\quad\quad\Sigma'\{x_1(x_1+\omega^2x_2+\omega^4x_3+\omega\, x_4+\omega^3x_5)\}\\
&\quad=\{\Sigma x^2+(\omega\ +\omega^4)\Sigma' x\, x_2+(\omega^2+\omega^3)\Sigma' x_1x_3\}\times\\
&\quad\quad\{\Sigma x^2+(\omega^2+\omega^3)\Sigma' x_1x_2+(\omega\ +\omega^4)\Sigma' x_1x_3\}\\
&\quad=(\Sigma x^2)^2-\Sigma x^2\Sigma x_1x_2-(\Sigma x_1x_2)^2+5\Sigma' x_1x_2\Sigma' x_1x_3,
\end{aligned}$$

which, by known relations among symmetric functions, may be readily put under the form exhibited at the foot of art. 6.

I may notice here that if, in place of $f(\omega)$, we had dealt with $f^m(\omega)$, that is

$$x_1^m+\omega x_2^m+\omega^2x_3^m+\omega^3x_4^m+\omega^4x_5^m,$$

we should have been led to

$$f^m(\omega).f^m(\omega^4)=\Sigma x^{2m}+(\omega\ +\omega^4)\Sigma' x_1^mx_2^m+(\omega^2+\omega^3)\Sigma' x_1^mx_3^m,$$

and

$$f^m(\omega^2).f^m(\omega^3)=\Sigma x^{2m}+(\omega^2+\omega^3)\Sigma' x_1^m x_2^m+(\omega+\omega^4)\Sigma' x_1^m x_3^m;$$

and therefore to

$$f^m(\omega).f^m(\omega^2).f^m(\omega^3).f^m(\omega^4)=(\Sigma x^{2m})^2-\Sigma x^{2m}\Sigma x_1^m x_2^m-(\Sigma x_1^m x_2^m)^2+5\Sigma' x_1^m x_2^m \Sigma' x_1^m x_3^m,$$

which, when $m=1$, coincides (as it ought to do) with the above result.

11. The expansion of $\Sigma' x_1^m x_2^m \Sigma' x_1^m x_3^m$ may, by Theorems I. and II., be effected thus,—

$$\begin{aligned}\Sigma' x_1^m x_2^m \Sigma' x_1^m x_3^m &= \Sigma' x_1^m x_2^m (x_1^m x_3^m + x_2^m x_4^m + x_3^m x_5^m + x_4^m x_1^m + x_5^m x_2^m)\\ &= \Sigma' (x_1^{2m} x_2^m x_3^m + x_1^{2m} x_3^m x_5^m + x_1^m x_2^m x_3^m x_4^m + x_1^{2m} x_2^m x_4^m + x_1^{2m} x_4^m x_5^m)\\ &= \Sigma' x_1^{2m} (x_2^m x_3^m + x_2^m x_4^m + x_3^m x_5^m + x_4^m x_5^m) + \Sigma x_1^m x_2^m x_3^m x_4^m;\end{aligned}$$

or since

$$\Sigma' x_1^{2m}(x_2^m x_3^m + x_2^m x_4^m + x_2^m x_5^m + x_3^m x_4^m + x_3^m x_5^m + x_4^m x_5^m)=\Sigma x_1^{2m} x_2^m x_3^m,$$

$$\therefore\quad \Sigma' x_1^m x_2^m \Sigma' x_1^m x_3^m = \Sigma x_1^{2m} x_2^m x_3^m + \Sigma x_1^m x_2^m x_3^m x_4^m - \Sigma' x_1^{2m}(x_2^m x_5^m + x_3^m x_4^m).$$

Consequently

$$\begin{aligned}f^m(\omega).f^m(\omega^2).f^m(\omega^3).f^m(\omega^4)&=(\Sigma x^{2m})^2-\Sigma x^{2m}\Sigma x_1^m x_2^m\\ &+5\Sigma x_1^{2m} x_2^m x_3^m-(\Sigma x_1^m x_2^m)^2+5\Sigma x_1^m x_2^m x_3^m x_4^m-5\Sigma' x_1^{2m}(x_2^m x_5^m+x_3^m x_4^m);\end{aligned}$$

which, making $m=1$, gives the resolvent product for quintics $\pi_4(x)$

$$=(\Sigma x^2)^2-\Sigma x^2 \Sigma x_1 x_2+5\Sigma x_1^2 x_2 x_3-(\Sigma x_1 x_2)^2+5\Sigma x_1 x_2 x_3 x_4-5\Sigma' x_1^2(x_2 x_5+x_3 x_4);$$

or, expressing the symmetric portion of this value in terms of the coefficients of the quintic, and writing λ for the unsymmetric portion $\Sigma' x_1^2(x_2x_5+x_3x_4)$, we have

$$\pi_4(x)=\frac{1}{a^4}(-15a^3e+5a^2bd+5a^2c^2-5ab^2c+b^4)-5\lambda.$$

I remark also that

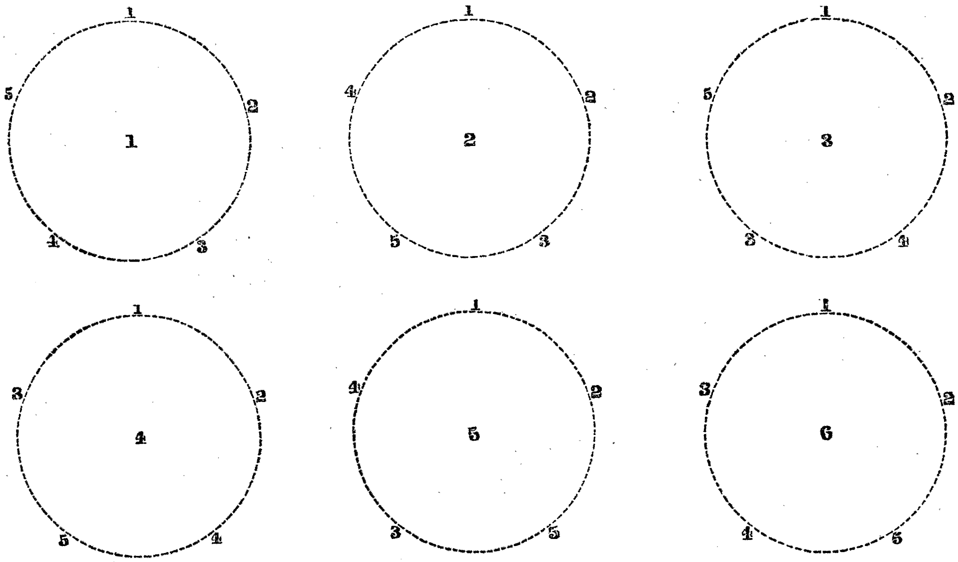
$$\tau\tau'=\Sigma' x_1 x_2 \Sigma' x_1 x_3=\Sigma x_1^2 x_2 x_3+\Sigma x_1 x_2 x_3 x_4-\lambda=\frac{1}{a^2}(-3ae+bd)-\lambda.$$

12. The number of unequal values of the several functions τ, τ', $\tau\tau'$, λ and $\pi_4(x)$ may be determined by the following considerations. Since the coefficients of the several terms in the expression for τ are equal, we may, in forming its values, regard one of the roots as fixed, while the others are permuted *inter se*. Thus we shall have 1.2.3.4 cycles, giving rise to 24 corresponding expressions for τ; but since, for the first cycle,

$$\tau=\Sigma' x_1 x_2=\Sigma' x_1 x_5,$$

and similar relations obtain for the other cycles, therefore these 24 expressions may be grouped in pairs, the members of each pair being equal. Hence τ has only 12 values. In the same way it may be shown that τ' has only 12 values. And since the several values of τ' may be referred to the same cycles which arise in the formation of the values of τ, and since also τ and τ' are complementary to each other (for their sum $=\Sigma x_1 x_2$, a one-valued function), it follows that $\tau\tau'$, and therefore also λ and $\pi_4(x)$, are six-valued functions. An independent proof that the resolvent product $\pi_4(x)$ is six-valued may be found in my original memoir, Section II., art. 15.

13. Fixing two of the roots, x_1 and x_2, and permuting the others, the following cycles arise, viz.:—



Six values of τ will then be included in the formula

$$\Sigma'_{1 \text{ to } 6}\, x_1x_2,$$

and the other values, which are in fact the corresponding values of τ', will be included in the formulæ,

$$\Sigma'_{1,\,2,\,3,\,5}\, x_1x_3 \text{ and } \Sigma'_{4,\,6}\, x_1x_4.$$

The values of λ are as follows:—

$$\lambda_1=\Sigma'_1x_1^2(x_2x_5+x_3x_4),\qquad \lambda_4=\Sigma'_4x_1^2(x_2x_3+x_4x_5),$$
$$\lambda_2=\Sigma'_2x_1^2(x_2x_4+x_3x_5),\qquad \lambda_5=\Sigma'_5x_1^2(x_2x_4+x_3x_5),$$
$$\lambda_3=\Sigma'_3x_1^2(x_2x_5+x_3x_4),\qquad \lambda_6=\Sigma'_6x_1^2(x_2x_3+x_4x_5).$$

And by Theorem III., art. 9,

$$\Sigma\lambda=\frac{5.6}{30}(1+1)\Sigma x_1^2x_2x_3=2\Sigma x_1^2x_2x_3,$$

a result which may be verified by actual development and summation. By Theorem II.

$$\begin{aligned}\lambda_1^2&=\Sigma'_1\{x_1^2(x_2x_5+x_3x_4)\Sigma'_1x_1^2(x_2x_5+x_3x_4)\}\\ &=\Sigma'_1[\,x_1^2(x_2x_5+x_3x_4)\{x_1^2(x_2x_5+x_3x_4)+x_2^2(x_3x_1+x_4x_5)\\ &\qquad+x_3^2(x_4x_2+x_5x_1)+x_4^2(x_5x_3+x_1x_2)+x_5^2(x_1x_4+x_2x_3)\}];\end{aligned}$$

or, multiplying out and reducing by the aid of Theorem I.*, we have

$$\begin{aligned}\lambda_1^2=\Sigma'_1\{&(x_1^4x_2^2x_5^2+x_1^4x_3^2x_4^2)+2(x_1^3x_2^3x_3x_5+x_1^3x_3^3x_4x_5)\\ &+2(x_1^3x_2^2x_3^2x_4+x_1^3x_2^2x_4^2x_5+x_1^3x_2x_3^2x_5^2+x_1^3x_3x_4^2x_5^2)\}\\ &+2\Sigma x_1^4x_2x_3x_4x_5+2\Sigma x_1^2x_2^2x_3^2x_4x_5;\end{aligned}$$

* In effecting reductions by the cyclical process, the greatest exponent is made the leading exponent.

and, therefore, by Theorem III.,

$$\Sigma\lambda^2=2\Sigma x_1^4x_2^2x_3^2+12\Sigma x_1^4x_2x_3x_4x_5+4\Sigma x_1^3x_2^3x_3x_4+4\Sigma x_1^3x_2^2x_3^2x_4+12\Sigma x_1^2x_2^2x_3^2x_4x_5.$$

Operating in the same way upon λ^2, we are led to the corresponding value of λ^3, and thence to $\Sigma\lambda^3$. Repeating the operation upon λ^3, we find $\Sigma\lambda^4$; and so on. But it may be remarked here that in dealing with the higher dimensioned functions, it will be found convenient to drop the subject x and to work only with exponents. Thus, following the first cycle and omitting, for convenience, the unit-suffix, we have

$$\begin{aligned}\lambda=\Sigma'\{(21001)+(20110)\}=&(21001)+(20110)\\ &+(12100)+(02011)\\ &+(01210)+(10201)\\ &+(00121)+(11020)\\ &+(10012)+(01102),\end{aligned}$$

and

$$\begin{aligned}\lambda=\Sigma'[&\{(42002)+(40220)\}+2\{(33101)+(30311)\}\\ &+2\{(32210)+(32021)+(31202)+(30122)\}]\\ &+8\Sigma 42^4+4\Sigma 3^321+2\Sigma 3^22^3+2\{\Sigma 41^4+\Sigma 2^31^2\}\lambda.\end{aligned}$$

And the passage to $\Sigma\lambda^3$ is easily effected. Carrying forward the calculation and collecting results, we have*

$$\begin{aligned}
\Sigma\lambda &= 2\Sigma 21^2,\\
\Sigma\lambda^2 &= 2\Sigma 42^2+12\Sigma 41^4+4\Sigma 3^21^2+4\Sigma 32^21+12\Sigma 2^31^2,\\
\Sigma\lambda^3 &= 2\Sigma 63^2+2\Sigma 62^21^2+3\Sigma 5421+6\Sigma 532^2\\
&\quad+8\Sigma 5321^2+6\Sigma 4^231+12\Sigma 4^221^2+12\Sigma 43^22\\
&\quad+8\Sigma 43^21^2+6\Sigma 432^21+48\Sigma 42^4+24\Sigma 3^321\\
&\quad+12\Sigma 3^22^3+2\{\Sigma 41^4+\Sigma 2^31^2\}\Sigma\lambda,\\
\Sigma\lambda^4 &= 2\Sigma 84^2+4\Sigma 83^21^2+12\Sigma 82^4+4\Sigma 7531\\
&\quad+4\Sigma 7432+16\Sigma 742^21+16\Sigma 73^221\\
&\quad+12\Sigma 6^22^2+4\Sigma 6541+20\Sigma 6531^2\\
&\quad+24\Sigma 652^21+12\Sigma 64^22+20\Sigma 64^21^2+24\Sigma 643^2\\
&\quad+28\Sigma 64321+52\Sigma 63^22^2+24\Sigma 5^242+24\Sigma 5^241^2\\
&\quad+48\Sigma 5^23^2+16\Sigma 5^2321+72\Sigma 5^22^3+44\Sigma 54^221
\end{aligned}$$

Thus if, in the expression $\Sigma' x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta x_5^\epsilon$, we suppose (*ex. gr.*) that, of the five exponents, γ is the greatest, then this function must be replaced by its equivalent $\Sigma' x_1^\gamma x_2^\delta x_3^\epsilon x_4^\alpha x_5^\beta$. Or, suppose that the greatest exponent (γ) is repeated, and that the function takes the form $\Sigma' x_1^\alpha x_2^\beta x_3^\gamma x_4^\gamma x^\epsilon$; then this must be replaced by $\Sigma' x_1^\gamma x_2^\gamma x_3^\epsilon x_4^\alpha x_5^\beta$; and so on. Following this method, the comparison of similar functions is greatly facilitated.

* Many of the details of calculation are given in the third section of my original memoir; but the results there exhibited belong to the quintic wanting in its second, third, and fifth term. The results exhibited in the text belong to the perfect form.

$$
\begin{aligned}
&+ 68\Sigma543^21 + 48\Sigma5432^2 + 36\Sigma53^32 + 144\Sigma4^4 \\
&+ 24\Sigma4^331 + 72\Sigma4^32^2 + 56\Sigma4^23^22 + 24\Sigma43^4 \\
&+ \{8\Sigma42^4 + 4\Sigma3^321 + 2\Sigma3^22^3\}\Sigma\lambda + 2\{\Sigma41^4 + \Sigma2^31^2\}\Sigma\lambda^2,
\end{aligned}
$$

$$
\begin{aligned}
\Sigma\lambda^5 = {} & 2\Sigma\overline{10}5^2 + 6\Sigma\overline{10}4^21^2 + 4\Sigma\overline{10}3^22^2 + 5\Sigma9641 \\
&+ 5\Sigma9542 + 14\Sigma95321 + 28\Sigma943^21 + 32\Sigma9432^2 \\
&+ 10\Sigma8732 + 5\Sigma8651 + 44\Sigma86321 + 36\Sigma85^21^2 \\
&+ 30\Sigma8543 + 39\Sigma85421 + 96\Sigma8532^2 + 96\Sigma84^231 \\
&+ 96\Sigma84^22^2 + 20\Sigma843^22 + 360\Sigma83^4 + 72\Sigma7^241^2 \\
&+ 180\Sigma7^22^3 + 30\Sigma7652 + 50\Sigma7643 + 54\Sigma76421 \\
&+ 136\Sigma763^21 + 48\Sigma7632^2 + 136\Sigma75^221 + 60\Sigma754^2 \\
&+ 118\Sigma75431 + 76\Sigma7542^2 + 164\Sigma753^22 + 36\Sigma74^31 \\
&+ 232\Sigma74^232 + 48\Sigma743^3 + 120\Sigma6^31^2 + 60\Sigma6^253 \\
&+ 96\Sigma6^2521 + 92\Sigma6^2431 + 336\Sigma6^242^2 + 132\Sigma6^23^22 \\
&+ 72\Sigma65^24 + 164\Sigma65^231 + 128\Sigma65^22^2 + 188\Sigma654^21 \\
&+ 172\Sigma65432 + 276\Sigma653^3 + 120\Sigma64^32 + 224\Sigma64^23^2 \\
&+ 108\Sigma5^341 + 348\Sigma5^332 + 184\Sigma5^24^22 + 104\Sigma5^243^2 \\
&+ 144\Sigma54^33 + \{2\Sigma82^4 + 4\Sigma63^22^2 + 12\Sigma5^22^3 \\
&+ 6\Sigma543^21 + 6\Sigma5432^2 + 6\Sigma53^32 + 4\Sigma4^331 \\
&+ 12\Sigma4^32^2 + 8\Sigma4^23^22 + 4\Sigma43^4\}\Sigma\lambda \\
&+ \{8\Sigma42^4 + 4\Sigma3^321 + 2\Sigma3^22^3\}\Sigma\lambda^2 \\
&+ 2\{\Sigma41^4 + \Sigma2^31^2\}\Sigma\lambda^3,
\end{aligned}
$$

$$
\begin{aligned}
\Sigma\lambda^6 = {} & 2\Sigma\overline{12}6^2 + 8\Sigma\overline{12}5^21^2 + 10\Sigma\overline{12}4^22^2 + 24\Sigma\overline{12}3^4 \\
&+ 6\Sigma\overline{11}751 + 6\Sigma\overline{11}652 + 22\Sigma\overline{11}6421 + 22\Sigma\overline{11}5431 \\
&+ 64\Sigma\overline{11}53^22 + 64\Sigma\overline{11}4^232 + 15\Sigma\overline{10}842 + 6\Sigma\overline{10}761 \\
&+ 30\Sigma\overline{10}7421 + 96\Sigma\overline{10}73^21 + 56\Sigma\overline{10}6^21^2 + 30\Sigma\overline{10}653 \\
&+ 24\Sigma\overline{10}6521 + 30\Sigma\overline{10}64^2 + 128\Sigma\overline{10}642^2 + 192\Sigma\overline{10}63^22 \\
&+ 96\Sigma\overline{10}5^231 + 208\Sigma\overline{10}5^22^2 + 192\Sigma\overline{10}54^21 + 158\Sigma\overline{10}5432 \\
&+ 264\Sigma\overline{10}4^23^2 + 40\Sigma9^23^2 + 56\Sigma9851^2 + 52\Sigma98421 \\
&+ 60\Sigma9832^2 + 30\Sigma9762 + 30\Sigma9753 + 104\Sigma97521 \\
&+ 120\Sigma974^2 + 82\Sigma97431 + 136\Sigma9742^2 + 252\Sigma973^22 \\
&+ 40\Sigma96^23 + 48\Sigma96^221 + 336\Sigma96531 + 340\Sigma9652^2 \\
&+ 488\Sigma964^21 + 270\Sigma96432 + 288\Sigma963^3 + 360\Sigma95^3 \\
&+ 16\Sigma95^241 + 284\Sigma95^232 + 680\Sigma954^22 + 584\Sigma9543^2 \\
&+ 168\Sigma94^33 + 30\Sigma8^262 + 120\Sigma8^253 + 320\Sigma8^2431
\end{aligned}
$$

$$
\begin{array}{lllll}
+120\Sigma 8^2 42^2 & +\ 192\Sigma 8^2 3^2 2 & +\ 20\Sigma 87^2 1^2 & +236\Sigma 87621 & \\
+120\Sigma 8754 & +\ 288\Sigma 87531 & +\ 380\Sigma 8752^2 & +240\Sigma 874^2 1 & \\
+486\Sigma 87432 & +\ 348\Sigma 873^3 & +\ 372\Sigma 86^2 4 & +304\Sigma 86^2 31 & \\
+200\Sigma 86^2 2^2 & +\ 364\Sigma 86541 & +\ 702\Sigma 86532 & +646\Sigma 864^2 2 & \\
+804\Sigma 8643^2 & +1188\Sigma 85^3 1 & +\ 684\Sigma 85^2 42 & +844\Sigma 85^2 3^2 & \\
+488\Sigma 854^2 3 & +1920\Sigma 84^4 & +\ 360\Sigma 7^3 3 & +400\Sigma 7^2 631 & \\
+552\Sigma 7^2 62^2 & +\ 264\Sigma 7^2 5^2 & +\ 776\Sigma 7^2 541 & +565\Sigma 7^2 532 & \\
+480\Sigma 7^2 4^2 2 & +1024\Sigma 7^2 43^2 & +\ 264\Sigma 76^2 5 & +564\Sigma 76^2 41 & \\
+707\Sigma 76^2 32 & +\ 332\Sigma 765^2 1 & +\ 942\Sigma 76542 & +740\Sigma 7653^2 & \\
+660\Sigma 764^2 3 & +\ 504\Sigma 75^3 2 & +\ 936\Sigma 75^2 43 & +420\Sigma 754^3 & \\
+744\Sigma 6^3 51 & +\ 444\Sigma 6^3 42 & +1380\Sigma 6^3 3^2 & +768\Sigma 6^2 5^2 2 & \\
+476\Sigma 6^2 543 & +1092\Sigma 6^2 4^3 & +\ 444\Sigma 65^3 3 & +228\Sigma 65^2 4^2 & \\
+168\Sigma 5^4 4 & +\ \{60\Sigma 83^4 & +\ 30\Sigma 7^2 2^3 & +\ 6\Sigma 753^2 2 & \\
+\ 6\Sigma 74^3 1 & +\ 14\Sigma 74^2 32 & +\ 8\Sigma 743^3 & +\ 20\Sigma 6^3 1^2 & +48\Sigma 6^2 42^2 \\
+\ 6\Sigma 6^2 3^2 2 & +\ 6\Sigma 65^2 31 & +\ 12\Sigma 65^2 2^2 + 12\Sigma 654^2 1 & +\ 10\Sigma 65432 & \\
+\ 46\Sigma 653^3 & +\ 20\Sigma 64^3 2 & +\ 32\Sigma 64^2 3^2 & +\ 18\Sigma 5^3 41 & +58\Sigma 5^3 32 \\
+\ 26\Sigma 5^2 4^2 2 & +\ 14\Sigma 5^2 43^2 & +\ 24\Sigma 54^3 3\}\Sigma\lambda & +\ \{2\Sigma 82^4 & \\
+\ 4\Sigma 63^2 2^2 & +\ 12\Sigma 5^2 2^3 & +\ 6\Sigma 543^2 1 & +\ 6\Sigma 5432^2 & +6\Sigma 53^3 2 \\
+\ 4\Sigma 4^3 31 & +\ 12\Sigma 4^3 2^2 & +\ 8\Sigma 4^2 3^2 2 & +\ 4\Sigma 43^4\}\Sigma\lambda^2 + \{8\Sigma 42^4 & \\
+\ 4\Sigma 3^3 21 & +\ 2\Sigma 3^2 2^3\}\Sigma\lambda^3 & +\ 2\{\Sigma 41^4 & +\ \Sigma 2^3 1^2\}\Sigma\lambda^4. &
\end{array}
$$

The bar placed over a number consisting of two digits indicates that it is a single exponent. *Ex. gr.* $\Sigma\overline{12}6^2$ represents $\Sigma x_1^{12}x_2^6x_3^6$, $\Sigma\overline{11}751$ represents $\Sigma x_1^{11}x_2^7x_3^5x_4$, and so on.

14. We may regard λ as the root of a sextic equation; and since the coefficients of any equation may be considered as known, when the sums of the powers of its roots are known, therefore the formulæ in the last article may be considered as giving implicitly the coefficients of the sextic in λ. The equation in λ being once obtained, we may, by a transformation linear in λ, deduce the sextic in θ or in $\tau\tau'$. But the high dimensions of the symmetric functions in x present practical (almost insuperable) difficulties in passing to the corresponding functions of the coefficients of the quintic. Mr. CAYLEY has suggested to me a more convenient method, which, however, I must leave its author to expound himself.

15. But if, in place of the complete quintic, we take any one of the trinomial forms

$$x^5+ex+f=0,$$
$$x^5+dx^2+f=0,$$
$$x^5+cx^3+f=0,$$
$$x^5+bx^4+f=0,$$

to which, by the method of Mr. JERRARD, Mr. COCKLE, or Professor SYLVESTER, the complete quintic may be reduced, then the foregoing formulæ may be made available. For, any symmetric function of the roots may be resolved into a function of the sums of the powers of the roots, and these sums, in any of the above cases, can of course be easily calculated. Thus, taking the second form,

$$x^5+dx^2+f=0,$$

which in many respects is the most convenient, we readily find

$$\Sigma 1=0,\ \Sigma 2=0,\ \Sigma 3=-3d,\ \Sigma 4=0,$$

and therefore

$$\Sigma\lambda=2\Sigma 21^2=(\Sigma 1)^2\Sigma 2-2\Sigma 1\Sigma 3-(\Sigma 2)^2+2\Sigma 4=0.$$

In like manner we obtain

$$\begin{aligned}
&\Sigma\lambda^2=4.5^2df,\\
&\Sigma\lambda^3=6d^4,\\
&\Sigma\lambda^4=4.5^4d^2f^2,\\
&\Sigma\lambda^5=158.5d^5f+5^6f^4,\\
&\Sigma\lambda^6=6d^8+4.5^6d^3f^3.
\end{aligned}$$

Therefore the equation in λ is

$$\lambda^6-2.5^2df\lambda^4-2d^4\lambda^3+5^4d^2f^2\lambda^2-(58d^5+5^5f^3)f\lambda+d^8=0;$$

and since, in this case (art. 11),

$$\pi_4(x)=-5\lambda,$$

the corresponding equation for the resolvent product is

$$\theta^6-2.5^4df\theta^4+2.5^3d^4\theta^3+5^8d^2f^2\theta^2+5^5(58d^5+5^5f^3)f\theta+5^6d^8=0,$$

where I have written θ for $\pi_4(x)$. This equation was first given by Mr. COCKLE. See his paper entitled "Researches in the Higher Algebra," printed in the second part of the fifteenth volume of the 'Manchester Memoirs.' Some interesting and curious transformations of the equation may be found in the same paper.

SECTION III.

The Symmetric Product for Quintics.

16. We know that, for the perfect form (art. 11),

$$a^4\theta=-15a^3e+5a^2bd+5a^2c^2-5ab^2c+b^4-5a^4\lambda.$$

And by definition (art. 1),

$$\Pi(x)=\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6.$$

It hence appears that the symmetric product Π is of twenty-four dimensions with respect to x. The partitions of twenty-four, for the quintic, are as follows:—

5^44
5^431
5^42^2
5^421^2
5^41^4

5^34^21
5^3432
5^3431^2
5^342^21
5^3421^3
5^341^5
5^33^3
5^33^221
$5^33^21^3$
5^332^3
$5^332^21^2$
5^3321^4
5^331^6
5^32^41
$5^32^31^3$
$5^32^21^5$
5^321^7
5^31^9

5^24^32
$5^24^31^2$
$5^24^23^2$
5^24^2321
$5^24^231^3$
$5^24^22^3$
$5^24^22^21^2$
$5^24^221^4$
$5^24^21^6$
5^243^31
$5^243^22^2$
$5^243^221^2$
$5^243^21^4$
5^2432^31
$5^2432^21^3$
5^24321^5
5^2431^7
5^242^5
$5^242^41^2$
$5^242^31^4$
$5^242^21^6$
5^2421^8
5^241^{10}
5^23^42
$5^23^41^2$
$5^23^32^21$
$5^23^321^3$
$5^23^31^5$
$5^23^22^4$
$5^23^22^31^2$
$5^23^22^21^4$
$5^23^221^6$
$5^23^21^8$
5^232^51
$5^232^41^3$
$5^232^31^5$
$5^232^21^7$
5^2321^9
5^231^{11}
5^22^7
$5^22^61^2$
$5^22^51^4$
$5^22^41^6$
$5^22^31^8$
$5^22^21^{10}$
5^221^{12}
5^21^{14}

54^43
54^421
54^41^3
54^33^21
54^332^2
54^3321^2
54^331^4
54^32^31
$54^32^21^3$
54^321^5
54^31^7
54^23^32
$54^23^31^2$
$54^23^22^21$
$54^23^221^3$
$54^23^21^5$
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5431^{12}
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$4^43^21^2$
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4^4321^3
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2^{12}
$2^{11}1^2$
$2^{10}1^4$
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2^71^{10}
2^61^{12}
2^51^{14}
2^41^{16}
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2^21^{20}
21^{22}

1^{24}

And the corresponding combinations (differently arranged) are—

$a^{19}ef^4$	$a^{16}c^4e^4$	$a^{14}b^4c^2e^4$	$a^{13}b^3c^3d^5$	$a^{11}b^7c^3d^2f$	$a^8b^{14}f^2$
$a^{18}bdf^4$	$a^{16}c^3d^3ef$	$a^{14}b^4cd^3ef$	$a^{13}b^2c^7df$	$a^{11}b^7c^3de^2$	$a^8b^{13}cef$
$a^{18}be^2f^3$	$a^{16}c^3d^2e^3$	$a^{14}b^4cd^2e^3$	$a^{13}b^2c^7e^2$	$a^{11}b^7c^2d^3e$	$a^8b^{13}d^2f$
$a^{18}c^2f^4$	$a^{16}c^2d^5f$	$a^{14}b^4d^5f$	$a^{13}b^2c^6d^2e$	$a^{11}b^7cd^5$	$a^8b^{13}de^2$
$a^{18}cdef^3$	$a^{16}c^2d^4e^2$	$a^{14}b^4d^4e^2$	$a^{13}b^2c^5d^4$	$a^{11}b^6c^5df$	$a^8b^{12}c^2df$
$a^{18}ce^3f^2$	$a^{16}cd^6e$	$a^{14}b^3c^4df^2$	$a^{13}bc^9f$	$a^{11}b^6c^5e^2$	$a^8b^{12}c^2e^2$
$a^{18}d^3f^3$	$a^{16}d^8$	$a^{14}b^3c^4e^2f$	$a^{13}bc^8de$	$a^{11}b^6c^4d^2e$	$a^8b^{12}cd^2e$
$a^{18}d^2e^2f^2$	$a^{15}b^5ef^3$	$a^{14}b^3c^3d^2ef$	$a^{13}bc^7d^3$	$a^{11}b^6c^3d^4$	$a^8b^{12}d^4$
$a^{18}de^4f$	$a^{15}b^4cdf^3$	$a^{14}b^3c^3de^3$	$a^{13}c^{10}e$	$a^{11}b^5c^7f$	$a^8b^{11}c^4f$
$a^{18}e^6$	$a^{15}b^4ce^2f^2$	$a^{14}b^3c^2d^4f$	$a^{13}c^9d^2$	$a^{11}b^5c^6de$	$a^8b^{11}c^3de$
$a^{17}b^2cf^4$	$a^{15}b^4d^2ef^2$	$a^{14}b^3c^2d^3e^2$	$a^{12}b^9f^3$	$a^{11}b^5c^5d^3$	$a^8b^{11}c^2d^3$
$a^{17}b^2def^3$	$a^{15}b^4de^3f$	$a^{14}b^3cd^5e$	$a^{12}b^8cef^2$	$a^{11}b^4c^8e$	$a^8b^{10}c^5e$
$a^{17}b^2e^3f^2$	$a^{15}b^4e^5$	$a^{14}b^3d^7$	$a^{12}b^8d^2f^2$	$a^{11}b^4c^7d^2$	$a^8b^{10}c^4d^2$
$a^{17}bc^2ef^3$	$a^{15}b^3c^3f^3$	$a^{14}b^2c^6f^2$	$a^{12}b^8de^2f$	$a^{11}b^3c^9d$	$a^8b^9c^6d$
$a^{17}bcd^2f^3$	$a^{15}b^3c^2def^2$	$a^{14}b^2c^5def$	$a^{12}b^8e^4$	$a^{11}b^2c^{11}$	$a^8b^8c^8$
$a^{17}bcde^2f^2$	$a^{15}b^3c^2e^3f$	$a^{14}b^2c^5e^3$	$a^{12}b^7c^2df^2$	$a^{10}b^{11}df^2$	$a^7b^{15}ef$
$a^{17}bce^4f$	$a^{15}b^3cd^3f^2$	$a^{14}b^2c^4d^3f$	$a^{12}b^7c^2e^2f$	$a^{10}b^{11}e^2f$	$a^7b^{14}cdf$
$a^{17}bd^3ef^2$	$a^{15}b^3cd^2e^2f$	$a^{14}b^2c^4d^2e^2$	$a^{12}b^7cd^2ef$	$a^{10}b^{10}c^2f^2$	$a^7b^{14}ce^2$
$a^{17}bd^2e^3f$	$a^{15}b^3cde^4$	$a^{14}b^2c^3d^4e$	$a^{12}b^7cde^3$	$a^{10}b^{10}cdef$	$a^7b^{14}d^2e$
$a^{17}bde^5$	$a^{15}b^3d^4ef$	$a^{14}b^2c^2d^6$	$a^{12}b^7d^4f$	$a^{10}b^{10}ce^3$	$a^7b^{13}c^3f$
$a^{17}c^3df^3$	$a^{15}b^3d^3e^3$	$a^{14}bc^7ef$	$a^{12}b^7d^3e^2$	$a^{10}b^{10}d^3f$	$a^7b^{13}c^2de$
$a^{17}c^3e^2f^2$	$a^{15}b^2c^4ef^2$	$a^{14}bc^6d^2f$	$a^{12}b^6c^4f^2$	$a^{10}b^{10}d^2e^2$	$a^7b^{13}cd^3$
$a^{17}c^2d^2ef^2$	$a^{15}b^2c^3d^2f^2$	$a^{14}bc^6de^2$	$a^{12}b^6c^3def$	$a^{10}b^9c^3ef$	$a^7b^{12}c^4e$
$a^{17}c^2de^3f$	$a^{15}b^2c^3de^2f$	$a^{14}bc^5d^3e$	$a^{12}b^6c^3e^3$	$a^{10}b^9c^2d^2f$	$a^7b^{12}c^3d^2$
$a^{17}c^2e^5$	$a^{15}b^2c^3e^4$	$a^{14}bc^4d^5$	$a^{12}b^6c^2d^3f$	$a^{10}b^9c^2de^2$	$a^7b^{11}c^5d$
$a^{17}cd^4f^2$	$a^{15}b^2c^2d^3ef$	$a^{14}c^8df$	$a^{12}b^6c^2d^2e^2$	$a^{10}b^9cd^3e$	$a^7b^{10}c^7$
$a^{17}cd^3e^2f$	$a^{15}b^2c^2d^2e^3$	$a^{14}c^8e^2$	$a^{12}b^6cd^4e$	$a^{10}b^9d^5$	$a^6b^{16}df$
$a^{17}cd^2e^4$	$a^{15}b^2cd^5f$	$a^{14}c^7d^2e$	$a^{12}b^6d^6$	$a^{10}b^8c^4df$	$a^6b^{16}e^2$
$a^{17}d^5ef$	$a^{15}b^2cd^4e^2$	$a^{14}c^6d^4$	$a^{12}b^5c^5ef$	$a^{10}b^8c^4e^2$	$a^6b^{15}c^2f$
$a^{17}d^4e^3$	$a^{15}b^2d^6e$	$a^{13}b^7cf^3$	$a^{12}b^5c^4d^2f$	$a^{10}b^8c^3d^2e$	$a^6b^{15}cde$
$a^{16}b^4f^4$	$a^{15}bc^5df^2$	$a^{13}b^7def^2$	$a^{12}b^5c^4de^2$	$a^{10}b^8c^2d^4$	$a^6b^{15}d^3$
$a^{16}b^3cef^3$	$a^{15}bc^5e^2f$	$a^{13}b^7e^3f$	$a^{12}b^5c^3d^3e$	$a^{10}b^7c^6f$	$a^6b^{14}c^3e$
$a^{16}b^3d^2f^3$	$a^{15}bc^4d^2ef$	$a^{13}b^6c^2ef^2$	$a^{12}b^5c^2d^5$	$a^{10}b^7c^5de$	$a^6b^{14}c^2d^2$
$a^{16}b^3de^2f^2$	$a^{15}bc^4de^3$	$a^{13}b^6cd^2f^2$	$a^{12}b^4c^6df$	$a^{10}b^7c^4d^3$	$a^6b^{13}c^4d$
$a^{16}b^3e^4f$	$a^{15}bc^3d^4f$	$a^{13}b^6cde^2f$	$a^{12}b^4c^6e^2$	$a^{10}b^6c^7e$	$a^6b^{12}c^6$
$a^{16}b^2c^2df^3$	$a^{15}bc^3d^3e^2$	$a^{13}b^6ce^4$	$a^{12}b^4c^5d^2e$	$a^{10}b^6c^6d^2$	$a^5b^{17}cf$
$a^{16}b^2c^2e^2f^2$	$a^{15}bc^2d^5e$	$a^{13}b^6d^3ef$	$a^{12}b^4c^4d^4$	$a^{10}b^5c^8d$	$a^5b^{17}de$
$a^{16}b^2cd^2ef^2$	$a^{15}bcd^7$	$a^{13}b^6d^2e^3$	$a^{12}b^3c^8f$	$a^{10}b^4c^{10}$	$a^5b^{16}c^2e$
$a^{16}b^2cde^3f$	$a^{15}c^7f^2$	$a^{13}b^5c^3df^2$	$a^{12}b^3c^7de$	$a^9b^{12}cf^2$	$a^5b^{16}cd^2$
$a^{16}b^2ce^5$	$a^{15}c^6def$	$a^{13}b^5c^3e^2f$	$a^{12}b^3c^6d^3$	$a^9b^{12}def$	$a^5b^{15}c^3d$
$a^{16}b^2d^4f^2$	$a^{15}c^6e^3$	$a^{13}b^5c^2d^2ef$	$a^{12}b^2c^9e$	$a^9b^{12}e^3$	$a^5b^{14}c^5$
$a^{16}b^2d^3e^2f$	$a^{15}c^5d^3f$	$a^{13}b^5c^2de^3$	$a^{12}b^2c^8d^2$	$a^9b^{11}c^2ef$	$a^4b^{19}f$
$a^{16}b^2d^2e^4$	$a^{15}c^5d^2e^2$	$a^{13}b^5cd^4f$	$a^{12}bc^{10}d$	$a^9b^{11}cd^2f$	$a^4b^{18}ce$
$a^{16}bc^4f^3$	$a^{15}c^4d^4e$	$a^{13}b^5cd^3e^2$	$a^{12}c^{12}$	$a^9b^{11}cde^2$	$a^4b^{18}d^2$
$a^{16}bc^3def^2$	$a^{15}c^3d^6$	$a^{13}b^5d^5e$	$a^{11}b^{10}ef^2$	$a^9b^{11}d^3e$	$a^4b^{17}c^2d$
$a^{16}bc^3e^3f$	$a^{14}b^6df^3$	$a^{13}b^4c^5f^2$	$a^{11}b^9cdf^2$	$a^9b^{10}c^3df$	$a^4b^{16}c^4$
$a^{16}bc^2d^3f^2$	$a^{14}b^6e^2f^2$	$a^{13}b^4c^4def$	$a^{11}b^9ce^2f$	$a^9b^{10}c^3e^2$	$a^3b^{20}e$
$a^{16}bc^2d^2e^2f$	$a^{14}b^5c^2f^3$	$a^{13}b^4c^4e^3$	$a^{11}b^9d^2ef$	$a^9b^{10}c^2d^2e$	$a^3b^{19}cd$
$a^{16}bc^2de^4$	$a^{14}b^5cdef^2$	$a^{13}b^4c^3d^3f$	$a^{11}b^9de^3$	$a\ b^{10}cd^4$	$a^3b^{18}c^3$
$a^{16}bcd^4ef$	$a^{14}b^5ce^3f$	$a^{13}b^4c^3d^2e^2$	$a^{11}b^8c^3f^2$	$a^9b^9c^5f$	$a^2b^{21}d$
$a^{16}bcd^3e^3$	$a^{14}b^5d^3f^2$	$a^{13}b^4c^2d^4e$	$a^{11}b^8c^2def$	$a^9b^9c^4de$	$a^2b^{20}c^2$
$a^{16}bd^6f$	$a^{14}b^5d^2e^2f$	$a^{13}b^4cd^6$	$a^{11}b^8c^2e^3$	$a^9b^9c^3d^3$	$ab^{22}c$
$a^{16}bd^5e^2$	$a^{14}b^5de^4$	$a^{13}b^3c^6ef$	$a^{11}b^8cd^3f$	$a^9b^8c^6e$	b^{24}
$a^{16}c^5ef^2$	$a^{14}b^4c^3ef^2$	$a^{13}b^3c^5d^2f$	$a^{11}b^8cd^2e^2$	$a^9b^8c^5d^2$	
$a^{16}c^4d^2f^2$	$a^{14}b^4c^2d^2f^2$	$a^{13}b^3c^5de^2$	$a^{11}b^8d^4e$	$a^9b^7c^7d$	
$a^{16}c^4de^2f$	$a^{14}b^4c^2de^2f$	$a^{13}b^3c^4d^3e$	$a^{11}b^7c^4ef$	$a^9b^6c^9$	

The direct calculation of Π would involve the prior calculation of the equation in λ. But by combining Eulerian with Lagrangian functions, and introducing the resolvent product, we may obtain the symmetric product without forming the sextic.

17. In order to facilitate operations it will be convenient to replace the coefficient of the first term by unity, and to suppose the quintic to be deprived of its second term; that is, in effect, to deal with the equation under the form

$$(1,\ 0,\ c,\ d,\ e,\ f\between x,\ 1)^5=0.$$

Then, if we assume

$$f(\omega^m)=x_1+\omega^m x_2+\omega^{2m}x_3+\omega^{3m}x_4+\omega^{4m}x_5=5\beta_m,$$

where, as in former articles, ω is an unreal fifth root of unity, the following relation will obtain, viz.

$$x_{r+1}=\omega^{4r}\beta_1+\omega^{3r}\beta_2+\omega^{2r}\beta_3+\omega^r\beta_4,$$

r being 0, 1, 2, 3, or 4 indifferently. Developing and reducing by the known properties of ω, we find

$$\begin{aligned}
\Sigma x^2&=10(\beta_1\beta_4+\beta_2\beta_3),\\
\Sigma x^3&=15(\beta_1^2\beta_3+\beta_2^2\beta_1+\beta_4^2\beta_2+\beta_3^2\beta_4),\\
\Sigma x^4&=30(\beta_1^2\beta_4^2+\beta_2^2\beta_3^2)+20(\beta_1^3\beta_2+\beta_2^3\beta_4+\beta_4^3\beta_3+\beta_3^3\beta_1)+120\beta_1\beta_2\beta_3\beta_4,\\
\Sigma x^5&=5\Sigma\beta^5+100(\beta_1^3\beta_3\beta_4+\beta_2^3\beta_1\beta_3+\beta_4^3\beta_2\beta_1+\beta_3^3\beta_4\beta_2)\\
&\qquad+150(\beta_1^2\beta_2^2\beta_4+\beta_2^2\beta_4^2\beta_3+\beta_4^2\beta_3^2\beta_1+\beta_3^2\beta_1^2\beta_2).
\end{aligned}$$

And by the method of the limiting equation or otherwise,

$$\Sigma x^2=-2c,\ \Sigma x^3=-3d,\ \Sigma x^4=2c^2-4e,\ \Sigma x^5=5cd-5f.$$

Whence by comparison with the above,

$$\begin{aligned}
-c&=5(\beta_1\beta_4+\beta_2\beta_3),\\
-d&=5(\beta_1^2\beta_3+\beta_2^2\beta_1+\beta_4^2\beta_2+\beta_3^2\beta_4),\\
-e&=5(\beta_1^3\beta_2+\beta_2^3\beta_4+\beta_4^3\beta_3+\beta_3^3\beta_1)+3\beta_1\beta_2\beta_3\beta_4-\tfrac{1}{5}c^2,\\
-f&=\Sigma\beta^5-10(\beta_1^3\beta_3\beta_4+\beta_2^3\beta_1\beta_3+\beta_4^3\beta_2\beta_1+\beta_3^3\beta_4\beta_2)+\tfrac{1}{5}cd,\\
&=\Sigma\beta^5+10(\beta_1^2\beta_2^2\beta_4+\beta_2^2\beta_4^2\beta_3+\beta_4^2\beta_3^2\beta_1+\beta_3^2\beta_1^2\beta_2)-\tfrac{1}{5}cd\,*.
\end{aligned}$$

18. If now we suppose one of the constituents, say β_4, to vanish, and eliminate the remaining ones, the effect will be the same as if we supposed the symmetric product, expressed as a function of the coefficients, to vanish. For

$$5^4\beta_1\beta_2\beta_3\beta_4=\theta;$$

* The formulæ here exhibited are EULER's, or rather EULER's as simplified by Mr. COCKLE. They indicate the connexion between the coefficients of the quintic and the constituents of its roots. Those constituents, by LAGRANGE's process, are expressed as rational functions of the roots. It is to be observed that EULER's functions are cyclical. In fact, applying Σ' to the cycle

$$\ldots 12431243\ldots$$

the relations may be written thus:—

$$-2c=5\Sigma'\beta_1\beta_4,\ -d=5\Sigma'\beta_1^2\beta_3,\ -e=5\Sigma'\beta_1^3\beta_2+3\beta_1\beta_2\beta_3\beta_4-\tfrac{1}{5}c^2,$$
$$-f=\Sigma\beta^5-10\Sigma'\beta_1^3\beta_3\beta_4+\tfrac{1}{5}cd=\Sigma\beta^5+10\Sigma'\beta_1^2\beta_2\beta_4-\tfrac{1}{5}cd.$$

And by the working properties of Σ',

$$cd=5^2\Sigma'\beta_1^2\beta_3(\beta_1\beta_4+\beta_2\beta_3)=5^2\Sigma'(\beta_1^3\beta_3\beta_4+\beta_1^2\beta_2^2\beta_4).$$

so that the evanescence of β involves the evanescence of θ, and consequently of Π. Making β_4 vanish, the equations at the foot of the last article become

$$\begin{aligned} -c &= 5\beta_2\beta_3, \\ -d &= 5(\beta_1^2\beta_3+\beta_1\beta_2^2), \\ -e &= 5(\beta_1^3\beta_2+\beta_1\beta_3^3)-\tfrac{1}{5}c^2, \\ -f &= \beta_1^5+\beta_2^5+\beta_3^5+2\beta_1\beta_2^2c+\tfrac{1}{5}cd. \end{aligned}$$

And the elimination of β_1, β_2, β_3 gives

$$\Pi=k\times$$

$$\left.\begin{array}{l} +\ 1\ c^{12} \\ -\ 16\ c^{10}e \\ +\ 2\ c^9d^2 \\ +\ 96\ c^8e^2 \\ -\ 9\ c^7d^2e \\ -\ 1\ c^6d^4 \\ -266\ c^6e^3 \\ -\ 19\ c^5d^2e^2 \\ +\ 23\ c^4d^4e \\ +336\ c^4e^4 \\ -\ 2\ c^3d^6 \\ -\ 58\ c^3d^2e^3 \\ +\ 14\ c^2d^4e^2 \\ -160\ c^2e^5 \\ -\ 7\ cd^6e \\ +\ 35\ cd^2e^4 \\ +\ 1\ d^8 \\ -\ 10\ d^4e^3 \\ +\ 25\ e^6 \end{array}\right| \left.\begin{array}{l} -\ 65\ c^8d \\ +\ 560\ c^6de \\ -\ 85\ c^5d^3 \\ -1340\ c^4de^2 \\ +\ 510\ c^3d^3e \\ -\ 70\ c^2d^5 \\ +\ 875\ c^2de^3 \\ -\ 175\ cd^3e^2 \\ +\ 50\ d^5e \\ -\ 250\ de^4 \end{array}\right| \left.\begin{array}{l} f-\ 10\ c^7 \\ \ \ +125\ c^5e \\ \ \ +150\ c^4d^2 \\ \ \ -500\ c^3e^2 \\ \ \ -750\ c^2d^2e \\ \ \ -125\ cd^4 \\ \ \ +625\ ce^3 \\ \ \ +625\ d^2e^2 \end{array}\right| \left.\begin{array}{l} f^2+\ 625c^3d \\ \ \ \ -3125cde \end{array}\right| f^3=0.$$

In order to determine k (a constant numerical factor, dropped in the course of calculation), let us take the particular equation

$$x^5+cx^3=0,$$

for which we have (art. 11)

$$\theta=5c^2,$$

and consequently

$$\Pi=5^6c^{12}.$$

Then since, in this case, the coefficients d, e, f severally vanish, the foregoing formula gives

$$\Pi=kc^{12};$$

and therefore

$$k=5^6*.$$

* The symmetric product for the quintic, deprived of its second term,

$$x^5-5Px^3-5Qx^2-5Rx+E=0,$$

was first calculated by Mr. COCKLE. See his paper "On Equations of the Fifth Degree," published in the Appendix to the Lady's and Gentleman's Diary for 1858. In the second section of my original memoir I have verified his result by an independent calculation and supplied the constant numerical multiplier. Mr. COCKLE presented the product in the form of a function of P, Q, S and E, S being given by

$$S=P^2+R.$$

Following the notation of my friend, I gave the product in the same form. But the passage to the

19. We have thus obtained the symmetric product for the quintic wanting in its second term; but it seems desirable on many grounds to calculate the product for the perfect form. A variety of methods of performing this calculation might be suggested. The reversal of the problem of transformation appears, at first sight, the most easy and practicable, and if in the foregoing expression for Π the following substitutions be made, viz.—

$$-\frac{2b^2}{5}+c \text{ for } c,$$

$$\frac{4b^3}{5^2}-\frac{3bc}{5}+d \text{ for } d,$$

$$-\frac{3b^4}{5^3}+\frac{3b^2}{5^2}-\frac{2bd}{5}+e \text{ for } e,$$

$$\frac{4b^5}{5^5}-\frac{b^3c}{5^3}+\frac{b^2d}{5^2}-\frac{be}{5}\ +f \text{ for } f,$$

the result will be the symmetric product for the complete quintic

$$(1, b, c, d, e, f \between x, 1)^5=0.$$

But it will be found on trial that this process, though apparently simple, does in point of fact involve prodigious labour. Mr. SAMUEL BILLS of Hawton, who kindly undertook to assist me in the calculation, communicated to me in the early part of the present year that portion of the expression into which f^3 enters. But the difficulties of the calculation and the want of means of verifying successive results have led him to abandon the work as impracticable. An equally effective and a much more expeditious process is supplied by the following considerations.

20. It occurred to Mr. COCKLE that the symmetric product for the perfect quintic

$$x^5-5Mx^4-5Px^3-5Qx^2-5Rx+E=0$$

---

corresponding expression in P, Q, R and E is easily effected, and I find that the result (as yet unpublished) of the transformation is—

$$\Pi=5^{14}\times$$

$+625\ P^{12}$	$+325\ P^8Q$	$E+2\ P^7$	$E^2+1\ P^3Q$	E^3
$+2000\ P^{10}R$	$+560\ P^6QR$	$+5\ P^5R$	$+1\ PQR$	
$-250\ P^9Q^2$	$-85\ P^5Q^3$	$+6\ P^4Q^2$		
$+2400\ P^8R^2$	$+268\ P^4QR^2$	$+4\ P^3R^2$		
$-225\ P^7Q^2R$	$-102\ P^3Q^3R$	$+6\ P^2Q^2R$		
$-25\ P^6Q^4$	$+14\ P^2Q^5$	$+1\ PQ^4$		
$+1330\ P^6R^3$	$+35\ P^2QR^3$	$+1\ PR^3$		
$+95\ P^5Q^2R^2$	$-7\ PQ^3R^2$	$+1\ Q^2R^2$		
$-115\ P^4Q^4R$	$+2\ Q^5R$			
$+336\ P^4R^4$	$+2\ QR^4$			
$+10\ P^3Q^6$				
$-58\ P^3Q^2R^3$				
$+14\ P^2Q^4R^2$				
$+32\ P^2R^5$				
$-7\ PQ^6R$				
$-7\ PQ^2R^4$				
$+1\ Q^8$				
$+2\ Q^4R^3$				
$+1\ R^6$				

is given by the expression

$$\pi - D\pi . M + D^2\pi . \frac{M^2}{1.2} - D^3\pi . \frac{M^3}{1.2.3} + \&c.,$$

where π is the symmetric product for the imperfect form treated of in the foot-note under paragraph 18, D is the differential symbol

$$\frac{d}{dx_1} + \frac{d}{dx_2} + \frac{d}{dx_3} + \frac{d}{dx_4} + \frac{d}{dx_5},$$

and the relations

$$DP = D(-\tfrac{1}{5}\Sigma x_1 x_2) = -\tfrac{4}{5}\Sigma x = -4M,$$

and others corresponding to them, hold. Mr. COCKLE had further noticed that $D\pi$ is to be regarded as free from M, a condition substantially equivalent to the expunging of the portion

$$5\partial_b + 4b\partial_c$$

of the operator ∇ which will be presently considered. Although he had not accurately completed this process of derivation, yet he had made a near approach to its completion, when Mr. CAYLEY, to whom as well as to myself Mr. COCKLE had communicated it, showed that the same results might be more immediately and conveniently obtained by means of the quantical calculus, and in so doing he incidentally corrected an oversight which I had already pointed out to Mr. COCKLE. In a letter under date September 28, 1859, Mr. CAYLEY called my attention to the circumstance that the several coefficients of the resolvent equation of the quintic are leading coefficients of a covariant. Mr. COCKLE had previously suggested that the symmetric product Π, or the last coefficient, was such a term. The test that a function of the roots may be such a term is that it is reduced to zero by the operation

$$\partial_{x1} + \partial_{x2} + \partial_{x3} + \partial_{x4} + \partial_{x5}.$$

It is clear that this is the case with respect to each factor of the product

$$\theta_1 = f(\omega).f(\omega^2).f(\omega^3).f(\omega^4) = 5^4\beta_1\beta_2\beta_3\beta_4.$$

Therefore it is also the case with the product itself; and since the like is true with respect to the other five values of θ, it is also true with respect to any symmetrical function of the six values. Consequently each coefficient of the sextic in θ is the leading coefficient of a covariant. At present, however, we have only to deal with the last coefficient, that is, the symmetric product.

21. Π is a seminvariant* reduced to zero by the operation

* The term "Seminvariant" is due to Mr. CAYLEY, who in a letter to me dated March 22, 1860, says, "The meaning is a function which is reduced to zero by one only of the operators which reduce to zero an invariant. It is in fact the leading coefficient of a covariant. It may also be defined as a function of the coefficients which is not altered by the substitution of $x+h$ for x." Defined as functions of the coefficients which are not altered by the substitution of $x+h$ for x, seminvariants are what Mr. COCKLE (who discussed such functions some years ago in the third and concluding volume of the 'Mathematician,' and more recently in some of the other journals referred to in the foot-note under the first paragraph of this paper) calls "critical functions." I may add that some years since Mr. COCKLE pointed out that the factors of the resolvent product, and, consequently, the product itself, are critical functions.

$$\nabla=5\partial_b+4b\partial_c+3c\partial_d+2d\partial_e+e\partial_f;$$

and if we write

$$\Pi=\Pi_0+\Pi_1 b+\Pi_2 b^2+\ldots+\Pi_s b^s,$$

Π_0 is known, being what Π becomes when the quintic is deprived of its second term, and Π_1, Π_2, ... Π_s may be found from it by means of the formulæ

$$\Pi_1=-\frac{1}{5}\ \nabla'\Pi_0,$$

$$\Pi_2=-\frac{1}{2.5}(\nabla'\Pi_1+4\partial_c\Pi_0),$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\Pi_s=-\frac{1}{5s}(\nabla'\Pi_{s-1}+4\partial_c\Pi_{s-2}),$$

where

$$\nabla'=\nabla-(5\partial_b+4b\partial_c)=3c\partial_d+2d\partial_e+e\partial_f.$$

Assume

$$\Pi_0=A_0+B_0 f+C_0 f^2+D_0 f^3,$$

and

$$\nabla''=\nabla'-e\partial f=3c\partial_d+2d\partial_e;$$

then

$$\begin{aligned}-5\Pi_1=\nabla'\Pi_0=\quad&\nabla''A_0+B_0 e\\&+f\ (\nabla''B_0+2C_0 e)\\&+f^2(\nabla''C_0+3D_0 e)\\&+f^3\ \nabla''D_0;\end{aligned}$$

and if we write

$$\Pi_1=A_1+B_1 f+C_1 f^2+D_1 f^3,$$

then will

$$\begin{aligned}-2.5\Pi_2=\nabla'\Pi_1+4\partial_c\Pi_0=\quad&\nabla''A_1+\ B_1 e+4\partial_c A_0\\&+f\ (\nabla''B_1+2C_1 e+4\partial_c B_0)\\&+f^2(\nabla''C_1+3D_1 e+4\partial_c C_0)\\&+f^3(\nabla''D_1\qquad\quad+4\partial_c D_0);\end{aligned}$$

so, in general, if

$$\Pi_t=A_t+B_t f+C_t f^2+D_t f^3,$$

we shall have

$$\begin{aligned}-5(t+1)\Pi_{t+1}=\nabla'\Pi_t+4\partial_c\Pi_{t-1}=\quad&\nabla''A_t+\ B_t e+4\partial_c A_{t-1}\\&+f\ (\nabla''B_t+2C_t e+4\partial_c B_{t-1})\\&+f^2(\nabla''C_t+3D_t e+4\partial_c C_{t-1})\\&+f^3(\nabla''D_t+4\partial_c D_{t-1}\qquad).\end{aligned}$$

These formulæ will enable us with comparative ease and great rapidity to derive the symmetric product for the complete quintic from that for the quintic wanting in its second term.

22. But it is noteworthy that the complete value of a seminvariant or critical function may always be deduced from the result obtained when *any* coefficient (excepting only a the first) vanishes. Thus, for the equation

$$(a, b, c, d, e, f \between x, 1)^5=0,$$

the operator is

$$\nabla=5a\partial_b+4b\partial_c+3c\partial_d+2d\partial_e+e\partial_f;$$

and if *ex. gr.* we write

$$\mathrm{I}=\mathrm{I}_0+\mathrm{I}_1 d+\mathrm{I}_2 d^2+\ldots$$

and suppose I_0 given, all the other terms can be found. For, in this case, let

$$\nabla'=\nabla-(3c\partial_d+2d\partial_e)=5a\partial_b+4b\partial_c+e\partial_f;$$

then

$$\nabla\mathrm{I}=\left\{\begin{array}{lll} \nabla'\mathrm{I}_0 & +d\nabla'\mathrm{I}_1+d^2\nabla\mathrm{I}_2+\ldots & \\ +3c(\mathrm{I}_1 & +2d\mathrm{I}_2+3d^2\mathrm{I}_3+\ldots) & \\ +2\,(d\partial_e\mathrm{I}_0+d^2\partial_e\mathrm{I}_1 & \qquad\qquad +\ldots) & \end{array}\right\}=0,$$

and therefore

$$3c\mathrm{I}_1=-\nabla'\mathrm{I}_0,$$

$$6c\mathrm{I}_2=-\nabla'\mathrm{I}_1-2\partial_e\mathrm{I}_0,$$

$$9c\mathrm{I}_3=-\nabla'\mathrm{I}_2-2\partial_e\mathrm{I}_1, \text{ \&c.},$$

which give I_1, I_2, I_3, &c. And it is easy to see how to obtain the corresponding formulæ for the other cases, viz. when the given function is free from either c, e, or f. The extension of the process to equations of other degrees than the fifth presents no difficulty.

23. In applying the formulæ in art. 21, it will be convenient to omit the constant numerical factor 5^6. The calculation may be thus conducted:—

Terms in Π_0.

A_0		B_0		C_0		D_0	
c^{12}	$+\ 1$	c^8d	$-\ 65$	c^7	$-\ 10$	c^3d	$+\ 625$
$c^{10}e$	$-\ 16$	c^6de	$+\ 560$	c^5e	$+\ 125$	cde	$-\ 3125$
c^9d^2	$+\ 2$	c^5d^3	$-\ 85$	c^4d^2	$+\ 150$		
c^8e^2	$+\ 96$	c^4de^2	$-\ 1340$	c^3e^2	$-\ 500$		
c^7d^2e	$-\ 9$	c^3d^3e	$+\ 510$	c^2d^2e	$-\ 750$		
c^6d^4	$-\ 1$	c^2d^5	$-\ 70$	cd^4	$-\ 125$		
c^6e^3	$-\ 266$	c^2de^3	$+\ 875$	ce^3	$+\ 625$		
$c^5d^2e^2$	$-\ 19$	cd^3e^2	$-\ 175$	d^2e^2	$+\ 625$		
c^4d^4e	$+\ 23$	d^5e	$+\ 50$				
c^4e^4	$+\ 336$	de^4	$-\ 250$				
c^3d^6	$-\ 2$						
$c^3d^2e^3$	$-\ 58$						
$c^2d^4e^2$	$+\ 14$						
c^2e^5	$-\ 160$						
cd^6e	$-\ 7$						
cd^2e^4	$+\ 35$						
d^8	$+\ 1$						
d^4e^3	$-\ 10$						
e^6	$+\ 25$						

Calculation of Π_1.

$\nabla'' A_0 + B_0 e$	$\underbrace{3c\partial_d \quad 2d\partial_e}_{A_0}$		e B_0	$-5A_1$	A_1
$c^{10}d$	$+12$	-32	$=$	-20	$+4$
c^8de	-54	$+384$	-65	$+265$	-53
c^7d^3	-12	-18		-30	$+6$
c^6de^2	-114	-1596	$+560$	-1150	$+230$
c^5d^3e	$+276$	-76	-85	$+115$	-23
c^4d^5	-36	$+46$		$+10$	-2
c^4de^3	-348	$+2688$	-1340	$+1000$	-200
$c^3d^3e^2$	$+168$	-348	$+510$	$+330$	-66
c^2d^5e	-126	$+56$	-70	-140	$+28$
c^2de^4	$+210$	-1600	$+875$	-515	$+103$
cd^7	$+24$	-14		$+10$	-2
cd^3e^3	-120	$+280$	-175	-15	$+3$
d^5e^2		-60	$+50$	-10	$+2$
de^5		$+300$	-250	$+50$	-10

$\nabla'' B_0 + 2C_0 e$	$\underbrace{3c\partial_d \quad 2d\partial_e}_{B_0}$		$2e$ C_0	$-5B_1$	B_1
c^9	-195		$=$	-195	$+39$
c^7e	$+1680$		-20	$+1660$	-332
c^6d^2	-765	$+1120$		$+355$	-71
c^5e^2	-4020		$+250$	-3770	$+754$
c^4d^2e	$+4590$	-5360	$+300$	-470	$+94$
c^3d^4	-1050	$+1020$		-30	$+6$
c^3e^3	$+2625$		-1000	$+1625$	-325
$c^2d^2e^2$	-1575	$+5250$	-1500	$+2175$	-435
cd^4e	$+750$	-700	-250	-200	$+40$
ce^4	-750		$+1250$	$+500$	-100
d^6		$+100$		$+100$	-20
d^2e^3		-2000	$+1250$	-750	$+150$

$\nabla'' C_0 + 3D_0 e$	$\underbrace{3c\partial_d \quad 2d\partial_e}_{C_0}$		$3e$ D_0	$-5C_1$	C_1
c^5d	$+900$	$+250$	$=$	$+1150$	-230
c^3de	-4500	-2000	$+1875$	-4625	$+925$
c^2d^3	-1500	-1500		-3000	$+600$
cde^2	$+3750$	$+3750$	-9375	-1875	$+375$
d^3e		$+2500$		$+2500$	-500

$\nabla'' D_0$	$\underbrace{3c\partial_d \quad 2d\partial_e}_{D_0}$		$-5D_1$	D_1
c^4	$+1875$	$=$	$+1875$	-375
c^2e	-9375		-9375	$+1875$
cd^2		-6250	-6250	$+1250$

Following the same process we obtain the successive developments of Π_2, Π_3, ... Π_{24}.

Then introducing the first coefficient (a) of the quintic and restoring the constant numerical factor (5^6), we have, for the complete form

$$(a,\ b,\ c,\ d,\ e,\ f\between x,\ 1)^5=0,$$

the symmetric product

$$\Pi=\frac{1}{a^{24}}\text{ multiplied into}$$

$5^8\times$

$-125\ a^{18}cdef^3$
$+25\ a^{18}ce^3f^2$
$+25\ a^{18}d^2e^2f^2$
$-10\ a^{18}de^4f$
$+1\ a^{18}e^6$

$+5^7\times$

$+250\ a^{17}b^2def^3$
$-50\ a^{17}b^2e^3f^2$
$+375\ a^{17}bc^2ef^3$
$+250\ a^{17}bcd^2f^3$
$+75\ a^{17}bcde^2f^2$
$-20\ a^{17}bce^4f$
$-100\ a^{17}bd^3ef^2$
$+30\ a^{17}bd^2e^3f$
$-2\ a^{17}bde^5$
$+125\ a^{17}c^3df^3$
$-100\ a^{17}c^3e^2f^2$
$-150\ a^{17}c^2d^2ef^2$
$+175\ a^{17}c^2de^3f$
$-32\ a^{17}c^2e^5$
$-25\ a^{17}cd^4f^2$
$-35\ a^{17}cd^3e^2f$
$+7\ a^{17}cd^2e^4$
$+10\ a^{17}d^5ef$
$-2\ a^{17}d^4e^3$

$+5^6\times$

$-1250\ a^{16}b^3cef^3$
$-500\ a^{16}b^3d^2f^3$
$-250\ a^{16}b^3de^2f^2$
$+60\ a^{16}b^3e^4f$
$-1875\ a^{16}b^2c^2df^3$
$-75\ a^{16}b^2c^2e^2f^2$
$+525\ a^{16}b^2cd^2ef^2$
$-785\ a^{16}b^2cde^3f$
$+141\ a^{16}b^2ce^5$
$+150\ a^{16}b^2d^4f^2$
$+80\ a^{16}b^2d^3e^2f$
$-19\ a^{16}b^2d^2e^4$
$-375\ a^{16}bc^4f^3$
$+925\ a^{16}bc^3def^2$
$-325\ a^{16}bc^3e^3f$
$+600\ a^{16}bc^2d^3f^2$
$-435\ a^{16}bc^2d^2e^2f$
$+103\ a^{16}bc^2de^4$
$+40\ a^{16}bcd^4ef$
$+3\ a^{16}bcd^3e^3$
$-20\ a^{16}bd^6f$
$+2\ a^{16}bd^5e^2$
$+125\ a^{16}c^5ef^2$
$+150\ a^{16}c^4d^2f^2$
$-1340\ a^{16}c^4de^2f$
$+336\ a^{16}c^4e^4$
$+510\ a^{16}c^3d^3ef$
$-58\ a^{16}c^3d^2e^3$
$-70\ a^{16}c^2d^5f$
$+14\ a^{16}c^2d^4e^2$
$-7\ a^{16}cd^6e$
$+1\ a^{16}d^8$

$+5^5\times$

$+1000\ a^{15}b^5ef^3$
$+5125\ a^{15}b^4cdf^3$
$+1275\ a^{15}b^4ce^2f^2$
$+100\ a^{15}b^4d^2ef^2$
$+790\ a^{15}b^4de^3f$
$-156\ a^{15}b^4e^5$
$+3875\ a^{15}b^3c^3f^3$
$-2775\ a^{15}b^3c^2def^2$
$+2135\ a^{15}b^3c^2e^3f$
$-2550\ a^{15}b^3cd^3f^2$
$+2140\ a^{15}b^3cd^2e^2f$
$-445\ a^{15}b^3cde^4$
$-260\ a^{15}b^3d^4ef$
$+20\ a^{15}b^3d^3e^3$
$-2075\ a^{15}b^2c^4ef^2$
$-4825\ a^{15}b^2c^3d^2f^2$
$+12075\ a^{15}b^2c^3de^2f$
$-2680\ a^{15}b^2c^3e^4$
$-3030\ a^{15}b^2c^2d^3ef$
$+140\ a^{15}b^2c^2d^2e^3$
$+420\ a^{15}b^2cd^5f$
$-100\ a^{15}b^2cd^4e^2$
$+20\ a^{15}b^2d^6e$
$-1150\ a^{15}bc^5df^2$
$+3770\ a^{15}bc^5e^2f$
$+470\ a^{15}bc^4d^2ef$
$-1000\ a^{15}bc^4de^3$
$+30\ a^{15}bc^3d^4f$
$-330\ a^{15}bc^3d^3e^2$
$+140\ a^{15}bc^2d^5e$
$-10\ a^{15}bcd^7$
$-50\ a^{15}c^7f^2$
$+2800\ a^{15}c^6def$
$-1330\ a^{15}c^6e^3$
$-425\ a^{15}c^5d^3f$
$-95\ a^{15}c^5d^2e^2$
$+115\ a^{15}c^4d^4e$
$-10\ a^{15}c^3d^6$

$+5^5\times$

$-750\ a^{14}b^6df^3$
$-270\ a^{14}b^6e^2f^2$
$-2475\ a^{14}b^5c^2f^3$
$+45\ a^{14}b^5cdef^2$
$-880\ a^{14}b^5ce^3f$
$+400\ a^{14}b^5d^3f^2$
$-540\ a^{14}b^5d^2e^2f$
$+102\ a^{14}b^5de^4$
$+1495\ a^{14}b^4c^3ef^2$
$+4320\ a^{14}b^4c^2d^2f^2$
$-8250\ a^{14}b^4c^2de^2f$
$+1568\ a^{14}b^4c^2e^4$
$+1195\ a^{14}b^4cd^3ef$
$+6\ a^{14}b^4cd^2e^3$
$-58\ a^{14}b^4d^5f$
$+27\ a^{14}b^4d^4e^2$
$+3740\ a^{14}b^3c^4df^2$
$-7165\ a^{14}b^3c^4e^2f$
$-1260\ a^{14}b^3c^3d^2ef$
$+1635\ a^{14}b^3c^3de^3$
$-40\ a^{14}b^3c^2d^4f$
$+490\ a^{14}b^3c^2d^3e^2$
$-100\ a^{14}b^3cd^5e$
$+485\ a^{14}b^2c^6f^2$
$-8280\ a^{14}b^2c^5def$
$+3115\ a^{14}b^2c^5e^3$
$+720\ a^{14}b^2c^4d^3f$
$+1040\ a^{14}b^2c^4d^2e^2$
$-265\ a^{14}b^2c^3d^4e$
$+5\ a^{14}b^2c^2d^6$
$-1660\ a^{14}bc^7ef$
$-355\ a^{14}bc^6d^2f$
$+1150\ a^{14}bc^6de^2$
$-115\ a^{14}bc^5d^3e$
$-10\ a^{14}bc^4d^5$
$-325\ a^{14}c^8df$
$+480\ a^{14}c^8e^2$
$-45\ a^{14}c^7d^2e$
$-5\ a^{14}c^6d^4$

$+5^5\times$

$+630\ a^{13}b^7cf^3$
$+90\ a^{13}b^7def^2$
$+116\ a^{13}b^7e^3f$
$-355\ a^{13}b^6c^2ef^2$
$-1275\ a^{13}b^6cd^2f^2$
$+2481\ a^{13}b^6cde^2f$
$-399\ a^{13}b^6ce^4$
$-114\ a^{13}b^6d^3ef$
$-10\ a^{13}b^6d^2e^3$
$-3550\ a^{13}b^5c^3df^2$
$+5456\ a^{13}b^5c^3e^2f$
$+1149\ a^{13}b^5c^2d^2ef$
$-958\ a^{13}b^5c^2de^3$
$-48\ a^{13}b^5cd^4f$
$-219\ a^{13}b^5cd^3e^2$
$+14\ a^{13}b^5d^5e$
$-1143\ a^{13}b^4c^5f^2$
$+9717\ a^{13}b^4c^4def$
$-3002\ a^{13}b^4c^4e^3$
$-426\ a^{13}b^4c^3d^3f$
$-1480\ a^{13}b^4c^3d^2e^2$
$+138\ a^{13}b^4c^2d^4e$
$+8\ a^{13}b^4cd^6$
$+4690\ a^{13}b^3c^6ef$
$+1240\ a^{13}b^3c^5d^2f$
$-2950\ a^{13}b^3c^5de^2$
$+40\ a^{13}b^3c^4d^3e$
$+40\ a^{13}b^3c^3d^5$
$+1585\ a^{13}b^2c^7df$
$-1715\ a^{13}b^2c^7e^2$
$-195\ a^{13}b^2c^6d^2e$
$+50\ a^{13}b^2c^5d^4$
$+195\ a^{13}bc^9f$
$-265\ a^{13}bc^8de$
$+30\ a^{13}bc^7d^3$
$-80\ a^{13}c^{10}e$
$+10\ a^{13}c^9d^2$

$+5^5\times$

$-56\ a^{12}b^9f^3$
$+17\ a^{12}b^8cef^2$
$+123\ a^{12}b^8d^2f^2$
$-270\ a^{12}b^8de^2f$
$+37\ a^{12}b^8e^4$
$+1456\ a^{12}b^7c^2df^2$
$-2063\ a^{12}b^7c^2e^2f$
$-444\ a^{12}b^7cd^2ef$
$+241\ a^{12}b^7cde^3$
$+12\ a^{12}b^7d^4f$
$+30\ a^{12}b^7d^3e^2$
$+1117\ a^{12}b^6c^4f^2$
$-5903\ a^{12}b^6c^3def$
$+1515\ a^{12}b^6c^3e^3$
$+113\ a^{12}b^6c^2d^3f$
$+811\ a^{12}b^6c^2d^2e^2$
$-13\ a^{12}b^6cd^4e$
$-2\ a^{12}b^6d^6$
$-5577\ a^{12}b^5c^5ef$
$-1616\ a^{12}b^5c^4d^2f$
$+3047\ a^{12}b^5c^4de^2$
$+162\ a^{12}b^5c^3d^3e$
$-36\ a^{12}b^5c^2d^5$
$-3060\ a^{12}b^4c^6df$
$+2609\ a^{12}b^4c^6e^2$
$+744\ a^{12}b^4c^5d^2e$
$-84\ a^{12}b^4c^4d^4$
$-785\ a^{12}b^3c^8f$
$+995\ a^{12}b^3c^7de$
$-70\ a^{12}b^3c^6d^3$
$+380\ a^{12}b^2c^9e$
$-10\ a^{12}b^2c^8d^2$
$+20\ a^{12}bc^{10}d$
$+5\ a^{12}c^{12}$

$+5^5\times$

$+2\ a^{11}b^{10}ef^2$
$-276\ a^{11}b^9cdf^2$
$+384\ a^{11}b^9ce^2f$
$+58\ a^{11}b^9d^2ef$
$-22\ a^{11}b^9de^3$
$-556\ a^{11}b^8c^3f^2$
$+1971\ a^{11}b^8c^2def$
$-421\ a^{11}b^8c^2e^3$
$-4\ a^{11}b^8cd^3f$
$-194\ a^{11}b^8cd^2e^2$
$-2\ a^{11}b^8d^4e$
$+3629\ a^{11}b^7c^4ef$
$+1047\ a^{11}b^7c^3d^2f$
$-1623\ a^{11}b^7c^3de^2$
$-147\ a^{11}b^7c^2d^3e$
$+10\ a^{11}b^7cd^5$
$+3143\ a^{11}b^6c^5df$
$-2206\ a^{11}b^6c^5e^2$
$-897\ a^{11}b^6c^4d^2e$
$+52\ a^{11}b^6c^3d^4$
$+1372\ a^{11}b^5c^7f$
$-1588\ a^{11}b^5c^6de$
$+48\ a^{11}b^5c^5d^3$
$-794\ a^{11}b^4c^8e$
$-52\ a^{11}b^4c^7d^2$
$-100\ a^{11}b^3c^9d$
$-30\ a^{11}b^2c^{11}$

$+5^4\times$

$+100\ a^{10}b^{11}df^2$
$-140\ a^{10}b^{11}e^2f$
$+750\ a^{10}b^{10}c^2f^2$
$-1710\ a^{10}b^{10}cdef$
$+305\ a^{10}b^{10}ce^3$
$-10\ a^{10}b^{10}d^3f$
$+85\ a^{10}b^{10}d^2e^2$
$-6985\ a^{10}b^9c^3ef$
$-1830\ a^{10}b^9c^2d^2f$
$+2355\ a^{10}b^9c^2de^2$
$+220\ a^{10}b^9cd^3e$

$-\ \ 4\ a^{10}b^{9}d^{5}$	$+3555\ a^{9}b^{10}c^{3}df$	$-8755\ a^{8}b^{11}c^{4}f$	$+5^{3}\times$	$+5^{2}\times$
$-9550\ a^{10}b^{8}c^{4}df$	$-1816\ a^{9}b^{10}c^{3}e^{2}$	$+6265\ a^{8}b^{11}c^{3}de$	$-\ 26\ a^{6}b^{16}df$	$-\ 8\ a^{4}b^{19}f$
$+5670\ a^{10}b^{8}c^{4}e^{2}$	$-\ 825\ a^{9}b^{10}c^{2}d^{2}e$	$+\ 270\ a^{8}b^{11}c^{2}d^{3}$	$+\ 9\ a^{6}b^{16}e^{2}$	$+\ 52\ a^{4}b^{18}ce$
$+2640\ a^{10}b^{8}c^{3}d^{2}e$	$+\ 2\ a^{9}b^{10}cd^{4}$	$+9620\ a^{8}b^{10}c^{5}e$	$-\ 394\ a^{6}b^{15}c^{2}f$	$+\ 6\ a^{4}b^{18}d^{2}$
$-\ 60\ a^{10}b^{8}c^{2}d^{4}$	$+4282\ a^{9}b^{9}c^{5}f$	$+2410\ a^{8}b^{10}c^{4}d^{2}$	$+\ 142\ a^{6}b^{15}cde$	$+\ 220\ a^{4}b^{17}c^{2}d$
$-6835\ a^{10}b^{7}c^{6}f$	$-3766\ a^{9}b^{9}c^{4}de$	$+5700\ a^{8}b^{9}c^{6}d$	$+\ 4\ a^{6}b^{15}d^{3}$	$+\ 690\ a^{4}b^{16}c^{4}$
$+7025\ a^{10}b^{7}c^{5}de$	$-\ 112\ a^{9}b^{9}c^{3}d^{3}$	$+3450\ a^{8}b^{8}c^{8}$	$+\ 792\ a^{6}b^{14}c^{3}e$	$+5^{2}\times$
$+\ 30\ a^{10}b^{7}c^{4}d^{3}$	$-3716\ a^{9}b^{8}c^{6}e$	$+5^{3}\times$	$+\ 196\ a^{6}b^{14}c^{2}d^{2}$	$-\ 2\ a^{3}b^{20}e$
$+4800\ a^{10}b^{6}c^{7}e$	$-\ 778\ a^{9}b^{8}c^{5}d^{2}$	$+\ 52\ a^{7}b^{15}ef$	$+1140\ a^{6}b^{13}c^{4}d$	$-\ 20\ a^{3}b^{19}cd$
$+\ 700\ a^{10}b^{6}c^{6}d^{2}$	$-1400\ a^{9}b^{7}c^{7}d$	$+\ 494\ a^{7}b^{14}cdf$	$+1345\ a^{6}b^{12}c^{6}$	$-\ 130\ a^{3}b^{18}c^{3}$
$+1100\ a^{10}b^{5}c^{8}d$	$-\ 650\ a^{9}b^{6}c^{9}$	$-\ 193\ a^{7}b^{14}ce^{2}$	$+5^{2}\times$	$+5\times$
$+\ 405\ a^{10}b^{4}c^{10}$	$+5^{3}\times$	$-\ 42\ a^{7}b^{14}d^{2}e$	$+\ 190\ a^{5}b^{17}cf$	$+\ 4\ a^{2}b^{21}d$
$+5^{4}\times$	$+\ 30\ a^{8}b^{14}f^{2}$	$+2339\ a^{7}b^{13}c^{3}f$	$-\ 34\ a^{5}b^{17}de$	$+\ 81\ a^{2}b^{20}c^{2}$
$-\ 105\ a^{9}b^{12}cf^{2}$	$-\ 990\ a^{8}b^{13}cef$	$-1265\ a^{7}b^{13}c^{2}de$	$-\ 596\ a^{5}b^{16}c^{2}e$	$+5\times$
$+\ 120\ a^{9}b^{12}def$	$-\ 120\ a^{8}b^{13}d^{2}f$	$-\ 54\ a^{7}b^{13}cd^{3}$	$-\ 118\ a^{5}b^{16}cd^{2}$	$(-\ 6\ ab^{22}c)$
$-\ 18\ a^{9}b^{12}e^{3}$	$+\ 108\ a^{8}b^{13}de^{2}$	$-3374\ a^{7}b^{12}c^{4}e$	$-1400\ a^{5}b^{15}c^{3}d$	$+1\times$
$+1590\ a^{9}b^{11}c^{2}ef$	$-3985\ a^{8}b^{12}c^{2}df$	$-\ 892\ a^{7}b^{12}c^{3}d^{2}$	$-2550\ a^{5}b^{14}c^{5}$	$(+1\ b^{24})$*
$+\ 330\ a^{9}b^{11}cd^{2}f$	$+1772\ a^{8}b^{12}c^{2}e^{2}$	$-3100\ a^{7}b^{11}c^{5}d$		
$-\ 354\ a^{9}b^{11}cde^{2}$	$+\ 657\ a^{8}b^{12}cd^{2}e$	$-2550\ a^{7}b^{10}c^{7}$		
$-\ 22\ a^{9}b^{11}d^{3}e$	$+\ 3\ a^{8}b^{12}d^{4}$			

24. A partial verification of this value is afforded by supposing the last two coefficients of the quintic, viz. e and f, or (what in effect is the same thing) two of its roots, say x_4 and x_5, to vanish. We have then (art. 11 *et seq.*)

$$\theta_1=\theta_2=\frac{1}{a^3}(\mathrm{a}a+5dx_2),$$

$$\theta_3=\theta_5=\frac{1}{a^3}(\mathrm{a}a+5dx_3),$$

$$\theta_4=\theta_6=\frac{1}{a^3}(\mathrm{a}a+5dx_1),$$

where I have for the moment written a in place of

$$5a^2bd+5a^2c^2-5ab^2c+b^4.$$

And, consequently, in this case the product is

$$\frac{1}{a^{24}}\times\text{the square of}$$

$-\ 5^{3}\ a^{8}d^{4}$	$-\ 2^{2}.5^{3}\ a^{5}b^{3}c^{3}d$	$-11.5^{2}\ a^{3}b^{6}c^{3}$
$+\ 5^{3}\ a^{7}bcd^{3}$	$-\ 3.5^{3}\ a^{5}b^{2}c^{5}$	$+\ 2.5\ a^{2}b^{9}d$
$+\ 5^{3}\ a^{7}c^{3}d^{2}$	$+\ 5^{2}\ a^{4}b^{6}d^{2}$	$+18.5\ a^{2}b^{8}c^{2}$
$+2.5^{3}\ a^{6}bc^{4}d$	$+14.5^{2}\ a^{4}b^{5}c^{2}d$	$-\ 3.5\ ab^{10}c$
$+\ 5^{3}\ a^{6}c^{6}$	$+18.5^{2}\ a^{4}b^{4}c^{4}$	$+\ 1\ b^{12}$
$-2^{2}.5^{2}\ a^{5}b^{4}cd^{2}$	$-\ 2^{2}.5^{2}\ a^{3}b^{7}cd$	

Developing this expression, all the terms in Π not affected by e or f will be verified.

* It is noticeable that the following combinations do not enter into the above expression for Π, viz.

$$a^{19}ef^4 \quad a^{18}be^2f^3 \quad a^{18}d^3f^3 \quad a^{16}b^4f^4$$
$$a^{18}bdf^4 \quad a^{18}c^2f^4 \quad a^{17}b^2cf^4 \quad a^{14}b^3d^7.$$

25. But a more complete verification is obtained by taking the sum of all the numerical coefficients that enter into the expression for Π. That sum is

$$-4761104,$$

as it ought to be. For the roots of

$$(1, 1, 1, 1, 1, 1 \between x, 1)^5=0$$

are

$$-1, \alpha, -\alpha, \alpha^2, -\alpha^2,$$

α being an unreal cube root of unity; and for this particular form we have

$$\theta_1=11-10\alpha, \quad \theta_3=\ \ 21+10\alpha, \quad \theta_5=1,$$
$$\theta_2=\ \ -29\ , \quad \theta_4=-\ \ 4+20\alpha, \quad \theta_6=-24-20\alpha;$$

or

$$\theta_1\theta_3=331, \quad \theta_2\theta_5=-29, \quad \theta_4\theta_6=496.$$

Consequently

$$\Pi=\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6=-4761104.$$

26. Another convenient verification is afforded by writing

$$a, \quad 5b, \quad 10c, \quad 10d, \quad 5e, f$$

for

$$a, \quad b, \quad c, \quad d, \quad e, f$$

respectively, when the sum of the numerical coefficients of the several powers of f should be zero*. And the transformed result will be worth having for its own sake, as belonging to Mr. CAYLEY's standard

$$(a, b, c, d, e, f \between x, 1)^5.$$

* Not only the sum of *all* the numerical coefficients, but the sum of the numerical coefficients of *each power of f*, is zero. The reason is, because the roots of

$$(1, 1, 1, 1, 1, f \between x, 1)^5=0$$

are included in the form

$$-1+\omega^m\sqrt[5]{1-f},$$

and therefore $\beta=0$. So that in this case the symmetric product vanishes identically.

In relation to the foregoing property, Mr. CAYLEY remarks as follows:—More generally the forms

$$(1, 0, 0, 0, 0, 1+f \between x, y)^5, \quad (1, 1, 1, 1, 1, f \between x', y')^5$$

are equivalent, the modulus of substitution being unity, as is at once seen by writing

$$x=x'+y'$$
$$y=\ \ \ \ y';$$

and the leading coefficients of any covariant of the same two forms respectively are therefore absolutely identical. That is, any seminvariant of the form

$$(a, b, c, d, e, f \between x, y)^5$$

will have the same value, whether we write therein

$$(a, b, c, d, e, f)=(1, 0, 0, 0, 0, 1+f),$$

or

$$(a, b, c, d, e, f)=(1, 1, 1, 1, 1, \ f\ );$$

whence in particular a seminvariant which vanishes upon writing therein $(b, c, d, e)=(0, 0, 0, 0)$ will also vanish upon writing therein $(a, b, c, d, e)=(1, 1, 1, 1, 1)$; that is, in such a seminvariant the sum of the numerical coefficients of *each power* of f is zero.

That result, arranging the terms according to descending powers of f, is

$$\Pi=\frac{5^{14}}{a^{24}}\times$$

$$\left(\begin{array}{rl|rl}
-4 & a^{18}cde & -48 & a^{16}bc^{4} \\
+4 & a^{17}b^{2}de & +8 & a^{15}b^{5}e \\
+12 & a^{17}bc^{2}e & +164 & a^{15}b^{4}cd \\
+16 & a^{17}bcd^{2} & +248 & a^{15}b^{3}c^{3} \\
+16 & a^{17}c^{3}d & -60 & a^{14}b^{6}d \\
-20 & a^{16}b^{3}ce & -396 & a^{14}b^{5}c^{2} \\
-16 & a^{16}b^{3}d^{2} & +252 & a^{13}b^{7}c \\
-120 & a^{16}b^{2}c^{2}d & -56 & a^{12}b^{9}
\end{array}\right)f^{3}.$$

$$+\left(\begin{array}{rl|rl|rl|rl|rl}
+2 & a^{18}ce^{3} & +168 & a^{16}b^{2}cd^{2}e & -1328 & a^{15}b^{2}c^{4}e & +6208 & a^{14}b^{2}c^{6} & +50 & a^{11}b^{10}e \\
+4 & a^{18}d^{2}e^{2} & +96 & a^{16}b^{2}d^{4} & -6176 & a^{15}b^{2}c^{3}d^{2} & +180 & a^{13}b^{7}de & -27600 & a^{11}b^{9}cd \\
-2 & a^{17}b^{2}e^{3} & +592 & a^{16}bc^{3}de & -2944 & a^{15}bc^{5}d & -1420 & a^{13}b^{6}c^{2}e & -111200 & a^{11}b^{8}c^{3} \\
+12 & a^{17}bcde^{2} & +768 & a^{16}bc^{2}d^{3} & -256 & a^{15}c^{7} & -10200 & a^{13}b^{6}cd^{2} & +5000 & a^{10}b^{11}d \\
-32 & a^{17}bd^{3}e & +160 & a^{16}c^{5}e & -54 & a^{14}b^{6}e^{2} & -56800 & a^{13}b^{5}c^{3}d & +75000 & a^{10}b^{10}c^{2} \\
-32 & a^{17}c^{3}e^{2} & +384 & a^{16}c^{4}d^{2} & +36 & a^{14}b^{5}cde & -36576 & a^{13}b^{4}c^{5} & -26250 & a^{9}b^{12}c \\
-96 & a^{17}c^{2}d^{2}e & +102 & a^{15}b^{4}ce^{2} & +640 & a^{14}b^{5}d^{3} & +170 & a^{12}b^{8}ce & +3750 & a^{8}b^{14} \\
-32 & a^{17}cd^{4} & +16 & a^{15}b^{4}d^{2}e & +2392 & a^{14}b^{4}c^{3}e & +2460 & a^{12}b^{8}d^{2} & & \\
-20 & a^{16}b^{3}de^{2} & -888 & a^{15}b^{3}c^{2}de & +13824 & a^{14}b^{4}c^{2}d^{2} & +58240 & a^{12}b^{7}c^{2}d & & \\
-12 & a^{16}b^{2}c^{2}e^{2} & -1632 & a^{15}b^{3}cd^{3} & +23936 & a^{14}b^{3}c^{4}d & +89360 & a^{12}b^{6}c^{4} & &
\end{array}\right)f^{2}.$$

$$+\left(\begin{array}{rl|rl|rl|rl|rl}
-4 & a^{18}de^{4} & -832 & a^{15}b^{3}d^{4}e & -212480 & a^{14}bc^{7}e & -2361200 & a^{12}b^{6}c^{3}de & -38200000 & a^{10}b^{8}c^{4}d \\
-8 & a^{17}bce^{4} & +38640 & a^{15}b^{2}c^{3}de^{2} & -90880 & a^{14}bc^{6}d^{2} & +90400 & a^{12}b^{6}c^{2}d^{3} & -54680000 & a^{10}b^{7}c^{6} \\
+24 & a^{17}bd^{2}e^{3} & -19392 & a^{15}b^{2}c^{2}d^{3}e & -166400 & a^{14}c^{8}d & -4461600 & a^{12}b^{5}c^{5}e & +150000 & a^{9}b^{12}de \\
+280 & a^{17}c^{2}de^{3} & +5376 & a^{15}b^{2}cd^{5} & +580 & a^{13}b^{7}e^{3} & -2585600 & a^{12}b^{5}c^{4}d^{2} & +3975000 & a^{9}b^{11}c^{2}e \\
-112 & a^{17}cd^{3}e^{2} & +24128 & a^{15}bc^{5}e^{2} & +49620 & a^{13}b^{6}cde^{2} & -9792000 & a^{12}b^{4}c^{6}d & +1650000 & a^{9}b^{11}cd^{2} \\
+64 & a^{17}d^{5}e & +6016 & a^{15}bc^{4}d^{2}e & -4560 & a^{13}b^{6}d^{3}e & -5024000 & a^{12}b^{3}c^{8} & +35550000 & a^{9}b^{10}c^{3}d \\
+12 & a^{16}b^{3}e^{4} & +768 & a^{15}bc^{3}d^{4} & +218240 & a^{13}b^{5}c^{3}e^{2} & +96000 & a^{11}b^{9}ce^{2} & +85640000 & a^{9}b^{9}c^{5} \\
-628 & a^{16}b^{2}cde^{3} & +71680 & a^{15}c^{6}de & +91920 & a^{13}b^{5}c^{2}d^{2}e & +29000 & a^{11}b^{9}d^{2}e & -1237500 & a^{8}b^{13}ce \\
+128 & a^{16}b^{2}d^{3}e^{2} & -21760 & a^{15}c^{5}d^{3} & -7680 & a^{13}b^{5}cd^{4} & +1971000 & a^{11}b^{8}c^{2}de & -300000 & a^{8}b^{13}d^{2} \\
-520 & a^{16}bc^{3}e^{3} & -1760 & a^{14}b^{5}ce^{3} & +1554720 & a^{13}b^{4}c^{4}de & -8000 & a^{11}b^{8}cd^{3} & -19925000 & a^{8}b^{12}c^{2}d \\
-1392 & a^{16}bc^{2}d^{2}e^{2} & -2160 & a^{14}b^{5}d^{2}e^{2} & -136320 & a^{13}b^{4}c^{3}d^{3} & +7258000 & a^{11}b^{7}c^{4}e & -87550000 & a^{8}b^{11}c^{4} \\
+256 & a^{16}bcd^{4}e & -66000 & a^{14}b^{4}c^{2}de^{2} & +1500800 & a^{13}b^{3}c^{6}e & +4188000 & a^{11}b^{7}c^{3}d^{2} & +162500 & a^{7}b^{15}e \\
-256 & a^{16}bd^{6} & +19120 & a^{14}b^{4}cd^{3}e & +793600 & a^{13}b^{3}c^{5}d^{2} & +25144000 & a^{11}b^{6}c^{5}d & +6175000 & a^{7}b^{14}cd \\
-8576 & a^{16}c^{4}de^{2} & -1856 & a^{14}b^{4}d^{5} & +2028800 & a^{13}b^{2}c^{7}d & +21952000 & a^{11}b^{5}c^{7} & +58475000 & a^{7}b^{13}c^{3} \\
+6528 & a^{16}c^{3}d^{3}e & -114640 & a^{14}b^{3}c^{4}e^{2} & +499200 & a^{13}bc^{9} & -17500 & a^{10}b^{11}e^{2} & -812500 & a^{6}b^{16}d \\
-1792 & a^{16}c^{2}d^{5} & -40320 & a^{14}b^{3}c^{3}d^{2}e & -13500 & a^{12}b^{8}de^{2} & -855000 & a^{10}b^{10}cde & -24625000 & a^{6}b^{15}c^{2} \\
+316 & a^{15}b^{4}de^{3} & -2560 & a^{14}b^{3}c^{2}d^{4} & -206300 & a^{12}b^{7}c^{2}e^{2} & -10000 & a^{10}b^{10}d^{3} & +5937500 & a^{5}b^{17}c \\
+1708 & a^{15}b^{3}c^{2}e^{3} & -529920 & a^{14}b^{2}c^{5}de & -88800 & a^{12}b^{7}cd^{2}e & -6985000 & a^{10}b^{9}c^{3}e & -625000 & a^{4}b^{19} \\
+3424 & a^{15}b^{3}cd^{2}e^{2} & +92160 & a^{14}b^{2}c^{4}d^{3} & +4800 & a^{12}b^{7}d^{4} & -3660000 & a^{10}b^{9}c^{2}d^{2} & &
\end{array}\right)f$$

+(

+ 1	$a^{18}e^6$	− 4720000	$a^{13}b^3c^5de^2$	+ 112000000	$a^{10}b^6c^6d^2$
− 4	$a^{17}bde^5$	+ 128000	$a^{13}b^3c^4d^3e$	+ 352000000	$a^{10}b^5c^8d$
− 128	$a^{17}c^2e^5$	+ 256000	$a^{13}b^3c^3d^5$	+ 259200000	$a^{10}b^4c^{10}$
+ 56	$a^{17}cd^2e^4$	− 5488000	$a^{13}b^2c^7e^2$	− 56250	$a^9b^{12}e^3$
− 32	$a^{17}d^4e^3$	− 1248000	$a^{13}b^2c^6d^2e$	− 4425000	$a^9b^{11}cde^2$
+ 282	$a^{16}b^2ce^5$	+ 640000	$a^{13}b^2c^5d^4$	− 550000	$a^9b^{11}d^3e$
− 76	$a^{16}b^2d^2e^4$	− 3392000	$a^{13}bc^8de$	− 45400000	$a^9b^{10}c^3e^2$
+ 824	$a^{16}bc^2de^4$	+ 768000	$a^{13}bc^7d^3$	− 41250000	$a^9b^{10}c^2d^2e$
+ 48	$a^{16}bcd^3e^3$	− 2048000	$a^{13}c^{10}e$	+ 200000	$a^9b^{10}cd^4$
+ 64	$a^{16}bd^5e^2$	+ 512000	$a^{13}c^9d^2$	− 376600000	$a^9b^9c^4de$
+ 5376	$a^{16}c^4e^4$	+ 4625	$a^{12}b^8e^4$	− 22400000	$a^9b^9c^3d^3$
− 1856	$a^{16}c^3d^2e^3$	+ 120500	$a^{12}b^7cde^3$	− 743200000	$a^9b^8c^6e$
+ 896	$a^{16}c^2d^4e^2$	+ 30000	$a^{12}b^7d^3e^2$	− 311200000	$a^9b^8c^5d^2$
− 896	$a^{16}cd^6e$	+ 1515000	$a^{12}b^6c^3e^3$	− 1120000000	$a^9b^7c^7d$
+ 256	$a^{16}d^8$	+ 1622000	$a^{12}b^6c^2d^2e^2$	− 1040000000	$a^9b^6c^9$
− 156	$a^{15}b^4e^5$	− 52000	$a^{12}b^6cd^4e$	+ 675000	$a^8b^{13}de^2$
− 1780	$a^{15}b^3cde^4$	− 16000	$a^{12}b^6d^6$	+ 22150000	$a^8b^{12}c^2e^2$
+ 160	$a^{15}b^3d^3e^3$	+ 12188000	$a^{12}b^5c^4de^2$	+ 16425000	$a^8b^{12}cd^2e$
− 21440	$a^{15}b^2c^3e^4$	+ 1296000	$a^{12}b^5c^3d^3e$	+ 150000	$a^8b^{12}d^4$
+ 2240	$a^{15}b^2c^2d^2e^3$	− 576000	$a^{12}b^5c^2d^5$	+ 313250000	$a^8b^{11}c^3de$
− 3200	$a^{15}b^2cd^4e^2$	+ 20872000	$a^{12}b^4c^6e^2$	+ 27000000	$a^8b^{11}c^2d^3$
+ 1280	$a^{15}b^2d^6e$	+ 11904000	$a^{12}b^4c^5d^2e$	+ 962000000	$a^8b^{10}c^5e$
− 32000	$a^{15}bc^4de^3$	− 2688000	$a^{12}b^4c^4d^4$	+ 482000000	$a^8b^{10}c^4d^2$
− 21120	$a^{15}bc^3d^3e^2$	+ 31840000	$a^{12}b^3c^7de$	+ 2280000000	$a^8b^9c^6d$
+ 17920	$a^{15}bc^2d^5e$	− 4480000	$a^{12}b^3c^6d^3$	+ 2760000000	$a^8b^8c^8$
− 2560	$a^{15}bcd^7$	+ 24320000	$a^{12}b^2c^9e$	− 6031250	$a^7b^{14}ce^2$
− 85120	$a^{15}c^6e^3$	− 1280000	$a^{12}b^2c^8d^2$	− 2625000	$a^7b^{14}d^2e$
− 12160	$a^{15}c^5d^2e^2$	+ 5120000	$a^{12}bc^{10}d$	− 158125000	$a^7b^{13}c^2de$
+ 29440	$a^{15}c^4d^4e$	+ 2560000	$a^{12}c^{12}$	− 13500000	$a^7b^{13}cd^3$
− 5120	$a^{15}c^3d^6$	− 27500	$a^{11}b^9de^3$	− 843500000	$a^7b^{12}c^4e$
+ 1020	$a^{14}b^5de^4$	− 1052500	$a^{11}b^8c^2e^3$	− 446000000	$a^7b^{12}c^3d^2$
+ 31360	$a^{14}b^4c^2e^4$	− 970000	$a^{11}b^8cd^2e^2$	− 3100000000	$a^7b^{11}c^5d$
+ 240	$a^{14}b^4cd^2e^3$	− 20000	$a^{11}b^8d^4e$	− 5100000000	$a^7b^{10}c^7$
+ 2160	$a^{14}b^4d^4e^2$	− 16230000	$a^{11}b^7c^3de^2$	+ 703125	$a^6b^{16}e^2$
+ 130800	$a^{14}b^3c^3de^3$	− 2940000	$a^{11}b^7c^2d^3e$	+ 44375000	$a^6b^{15}cde$
+ 78400	$a^{14}b^3c^2d^3e^2$	+ 400000	$a^{11}b^7cd^5$	+ 2500000	$a^6b^{15}d^3$
− 32000	$a^{14}b^3cd^5e$	− 44120000	$a^{11}b^6c^5e^2$	+ 495000000	$a^6b^{14}c^3e$
+ 498400	$a^{14}b^2c^5e^3$	− 35880000	$a^{11}b^6c^4d^2e$	+ 245000000	$a^6b^{14}c^2d^2$
+ 332800	$a^{14}b^2c^4d^2e^2$	+ 4160000	$a^{11}b^6c^3d^4$	+ 2850000000	$a^6b^{13}c^4d$
− 169600	$a^{14}b^2c^3d^4e$	− 127040000	$a^{11}b^5c^6de$	+ 6725000000	$a^6b^{12}c^6$
+ 6400	$a^{14}b^2c^2d^6$	+ 7680000	$a^{11}b^5c^5d^3$	− 5312500	$a^5b^{17}de$
+ 736000	$a^{14}bc^6de^2$	− 127040000	$a^{11}b^4c^8e$	− 186250000	$a^5b^{16}c^2e$
− 147200	$a^{14}bc^5d^3e$	− 16640000	$a^{11}b^4c^7d^2$	− 73750000	$a^5b^{16}cd^2$
− 25600	$a^{14}bc^4d^5$	− 64000000	$a^{11}b^3c^9d$	− 1750000000	$a^5b^{15}c^3d$
+ 614400	$a^{14}c^8e^2$	− 38400000	$a^{11}b^2c^{11}$	− 6375000000	$a^5b^{14}c^5$
− 115200	$a^{14}c^7d^2e$	+ 381250	$a^{10}b^{10}ce^3$	+ 40625000	$a^4b^{18}ce$
− 25600	$a^{14}c^6d^4$	+ 212500	$a^{10}b^{10}d^2e^2$	+ 9375000	$a^4b^{18}d^2$
− 19950	$a^{13}b^6ce^4$	+ 11775000	$a^{10}b^9c^2de^2$	+ 687500000	$a^4b^{17}c^2d$
− 1000	$a^{13}b^6d^2e^3$	+ 2200000	$a^{10}b^9cd^3e$	+ 4312500000	$a^4b^{16}c^4$
− 191600	$a^{13}b^5c^2de^3$	− 80000	$a^{10}b^9d^5$	− 3906250	$a^3b^{20}e$
− 87600	$a^{13}b^5cd^3e^2$	+ 56700000	$a^{10}b^8c^4e^2$	− 156250000	$a^3b^{19}cd$
+ 11200	$a^{13}b^5d^5e$	+ 52800000	$a^{10}b^8c^3d^2e$	− 2031250000	$a^3b^{18}c^3$
− 1200800	$a^{13}b^4c^4e^3$	− 2400000	$a^{10}b^8c^2d^4$	+ 15625000	$a^2b^{21}d$
− 1184000	$a^{13}b^4c^3d^2e^2$	+ 281000000	$a^{10}b^7c^5de$	+ 632812500	$a^2b^{20}c^2$
+ 220800	$a^{13}b^4c^2d^4e$	+ 2400000	$a^{10}b^7c^4d^3$	− 117187500	$ab^{22}c$
+ 25600	$a^{13}b^4cd^6$	+ 384000000	$a^{10}b^6c^7e$	+ 9765625	b^{24}

)